Large Language Models

Scaling Laws

ELL881 · AIL821

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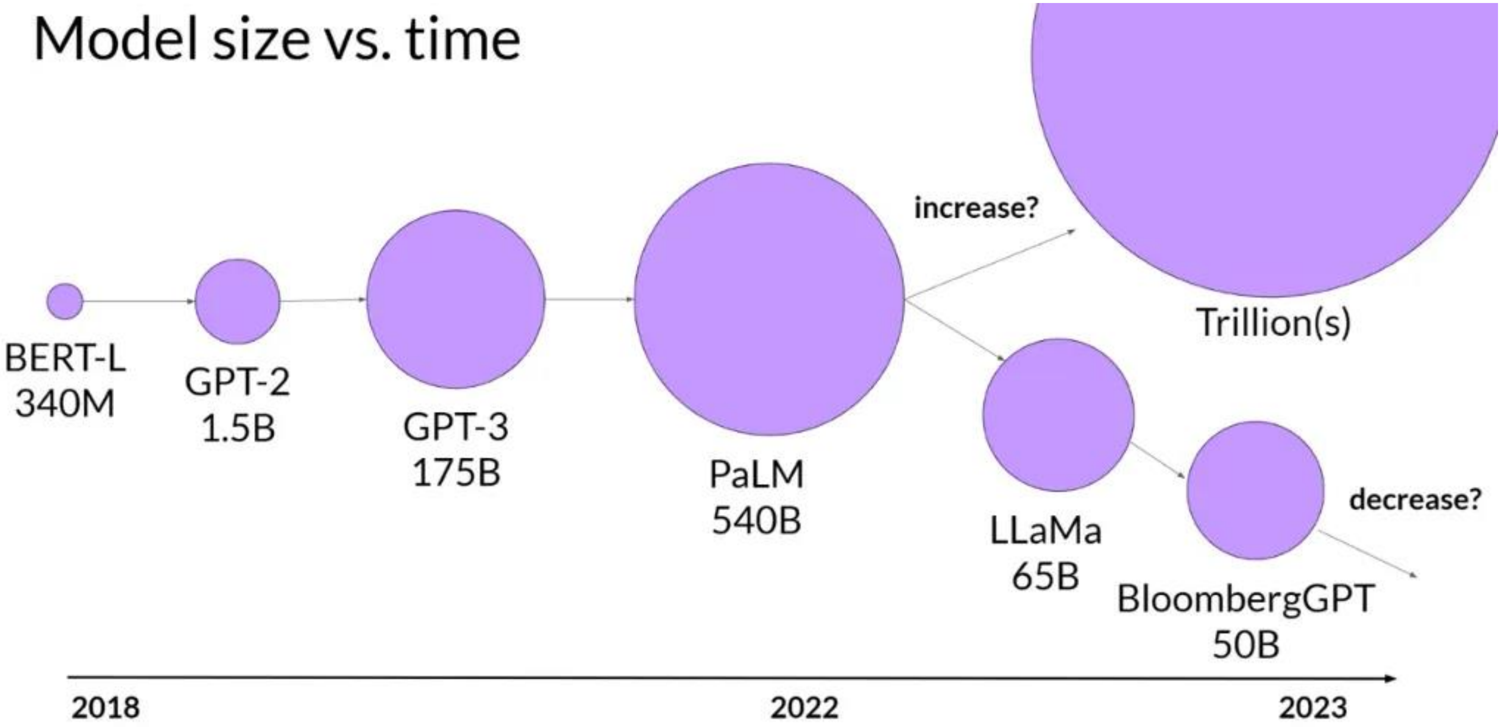
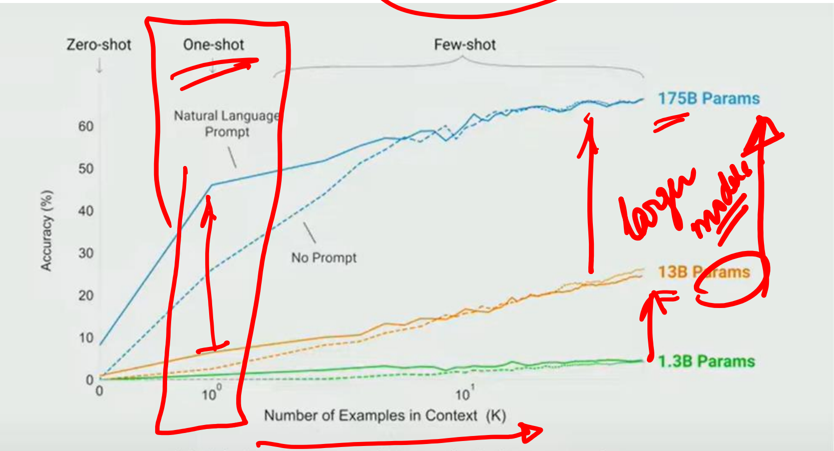
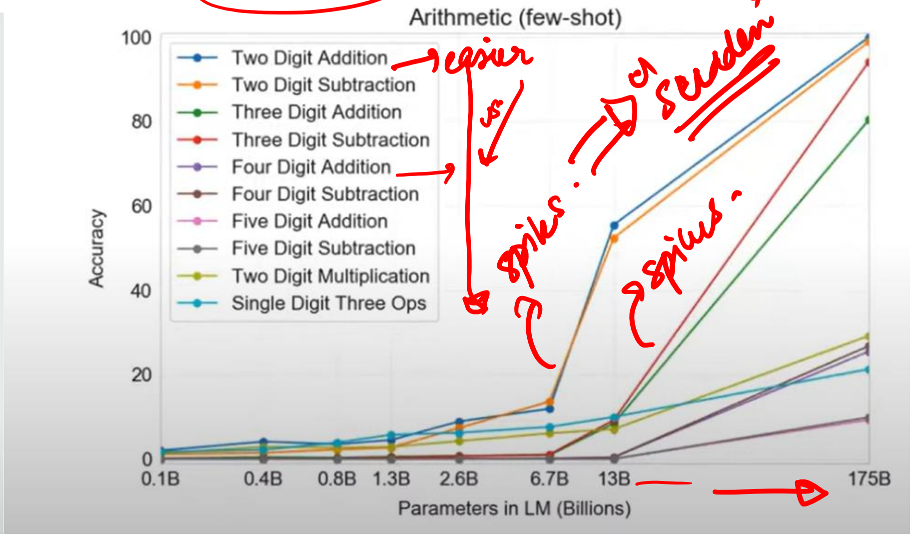

Model size vs. time
increase?
Trillion(s)
BERT-L
340M
GPT-2
1.5B
GPT-3
175B
PaLM
540B
LLaMa
65B
decrease?
BloombergGPT
50B
2018
2022
2023

“*Emergent*” abilities in LLM

LLMs are few-shot learners (*and larger is better!*)

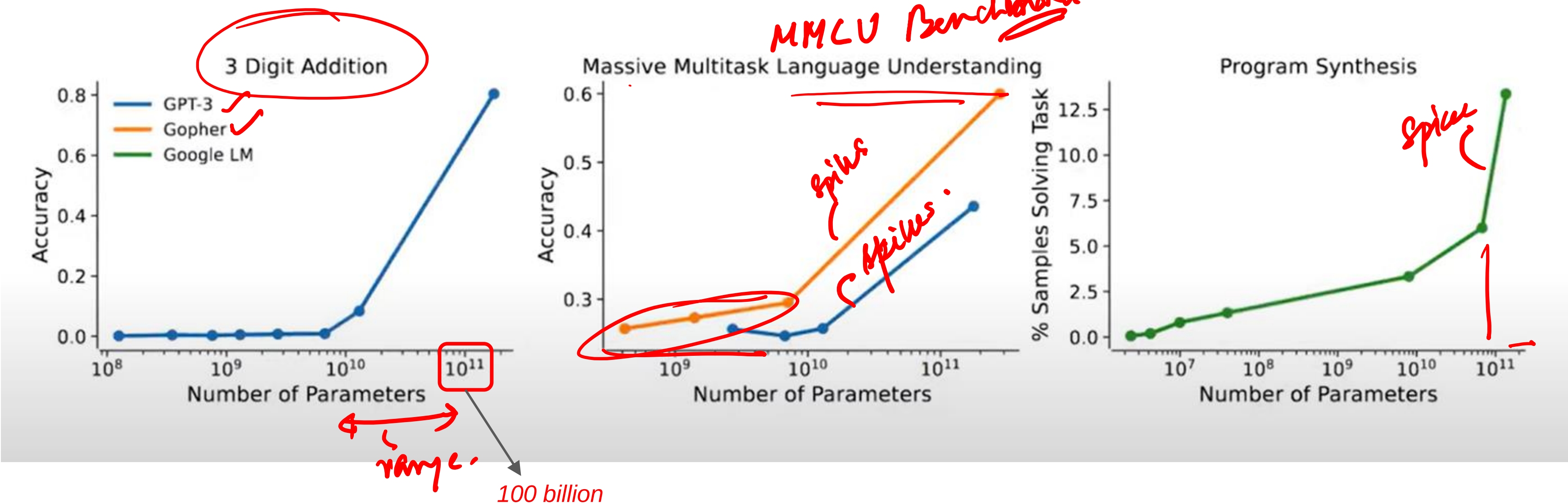


Task is to remove random symbols from a word.



LLMs are few-shot learners (*and larger is better!*)

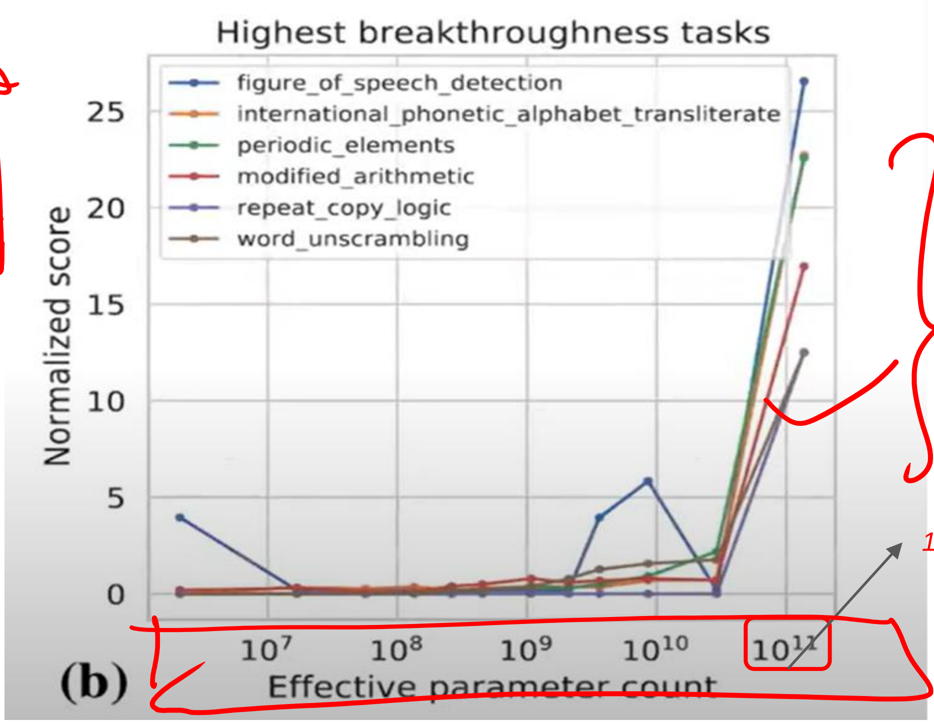
MMLU Benchmark.



100 billion

LLMs are few-shot learners (*and larger is better!*)

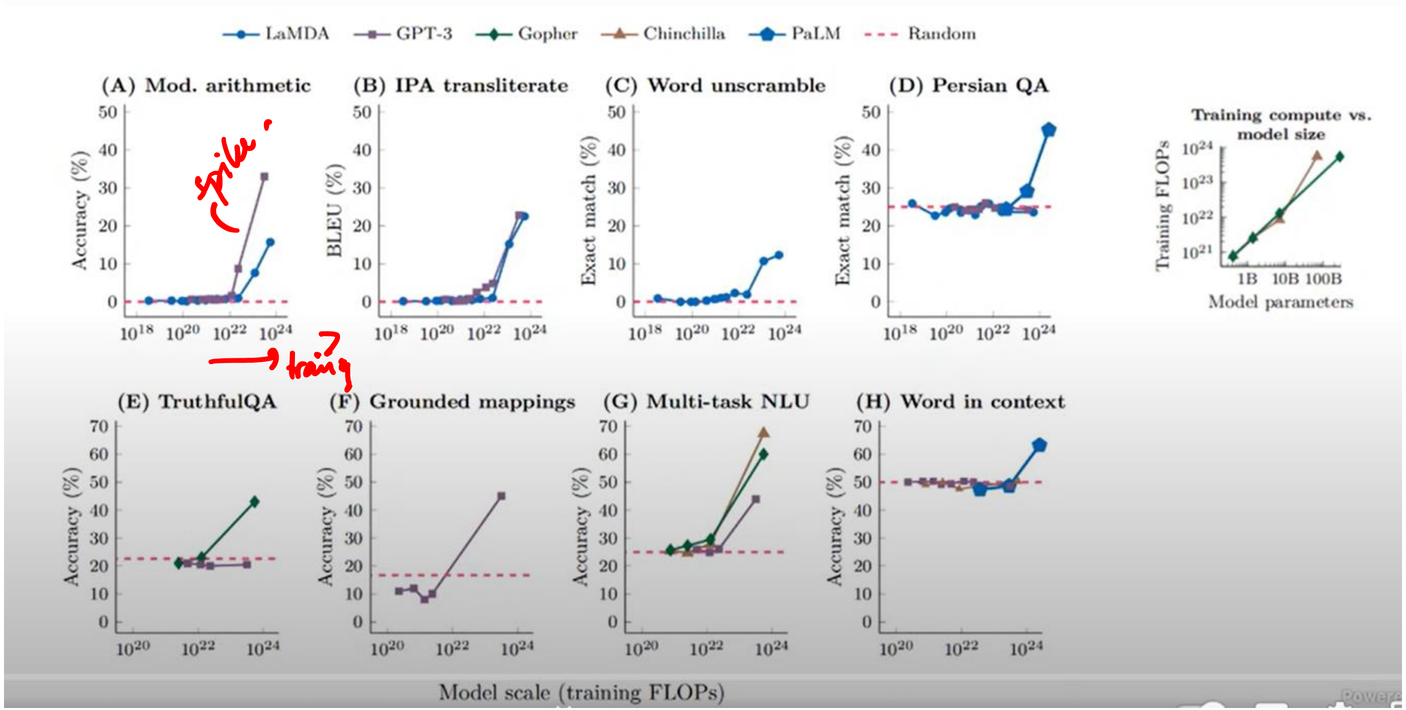
NLP tasks' ↑



Spikes.

100 billion

LLMs are few-shot learners (*more training is better too!*)



Google BIG-BENCH benchmark

"emergent" "ability".

"spike"

most In-context learning (ICL)

Consider a single Model Family e.g. PaLM

Let x_n be the scale of one family member e.g. PaLM-540B

Let y_n be the family member's score on some Task and Metric

Sort the pairs (x_n, y_n) by model scale x_n, smallest to largest

particular size of a given model.

expectation → +ve

$$\text{Emergence Score}\left(\left\{(x_n, y_n)\right\}_{n=1}^{N}\right) \stackrel{\text{def}}{=} \frac{\text{sign}(\arg\max_i y_i - \arg\min_i y_i)(\max_i y_i - \min_i y_i)}{\sqrt{\text{Median}(\{(y_i - y_{i-1})^2\}_i)}}$$

measure to quantify the spike.

normalizing

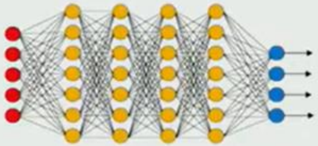
score given to the prediction.

LLMs "*seems*" to get more intelligent with the following:

NT: scale *w/o bottleneck*

Amount of training data

(I)



Model size (parameters)

(II)

Amount of compute (or time)

(III)

α x Model size x Training data = Training compute

$\boxed{\sim 6}$ FLOPs (Floating-point Operations)

Recap on Parameter size & FLOPs

context length [input sequence].

Operation	Parameters	FLOPs per Token
Embed	$(n_{\text{vocab}} + n_{\text{ctx}})\, d_{\text{model}}$	$4d_{\text{model}}$
Attention: QKV	$n_{\text{layer}} d_{\text{model}} 3d_{\text{attn}}$	$2n_{\text{layer}} d_{\text{model}} 3d_{\text{attn}}$
Attention: Mask	—	$2n_{\text{layer}} n_{\text{ctx}} d_{\text{attn}}$
Attention: Project	$n_{\text{layer}} d_{\text{attn}} d_{\text{model}}$	$2n_{\text{layer}} d_{\text{attn}} d_{\text{embd}}$
Feedforward	$n_{\text{layer}} 2d_{\text{model}} d_{\text{ff}}$	$2n_{\text{layer}} 2d_{\text{model}} d_{\text{ff}}$
De-embed	—	$2d_{\text{model}} n_{\text{vocab}}$
Total (Non-Embedding)	$N = 2d_{\text{model}} n_{\text{layer}} (2d_{\text{attn}} + d_{\text{ff}})$	$C_{\text{forward}} = 2N + 2n_{\text{layer}} n_{\text{ctx}} d_{\text{attn}}$

dim of FFN
sandwiched between 2 attn layers.

one PF-day $= 10^{15} \times 24 \times 3600 = \boxed{8.64 \times 10^{19}}$ floating point operations.

Emergent abilities are *unpredictable*

If that’s true, we never get to know the following:

- *Which* abilities (and *when*) exactly will emerge?
- *What* controls the trigger?
- Can we make desirable abilities to emerge *faster*?
- Can we make undesirable abilities to be suppressed?

Is the value of scaling laws only in predicting?

- How much *return* for a given compute (resource) budget?

- How to *allocate* the compute budget - model size vs. dataset size?

Can there be a curve that fits “emergence”? *Intuition*

Input: $x_1 \; ... x_n \; \sim N(\mu, \sigma^2)$

Task: estimate the average as $\hat{\mu} = \frac{\sum_i x_i}{n}$

What's the error? By standard arguments..

$$\boxed{E[(\hat{\mu} - \mu)^2]} = \boxed{\frac{\sigma^2}{n}}$$

This is a scaling law!!

$$\boxed{\log(Error) = -\log n + 2 \log \sigma}$$

More generally, any polynomial rate $\boxed{1/n^{\alpha}}$ is a scaling law

power law

Source: CS-324, Stanford University

What about fitting "emergence" in *non-parametric* setting

Neural nets can approximate arbitrary functions. Lets turn that into an example.

Input: $x_1 \ldots x_n$ uniform in 2D unit box. $y_i = f(x_i) + N(0,1)$

Task: estimate f(x)

Approach: cut up the 2D space into boxes with length $n^{-\frac{1}{4}}$, average in each box

$\frac{1}{(\sqrt{n})^2}$

What's our estimation error?

Informally, we have $\sqrt{n}$ boxes, each box gets $\sqrt{n}$ samples.

$$Error \approx \frac{1}{\sqrt{n}} + (other\ smoothness\ terms)$$

$\frac{1}{\sqrt[d]{n}}$

In d-dimensions, this becomes $Error = n^{-1/d}$ - **This means scaling is** $\boldsymbol{y = -\frac{1}{d}x + C}$

Takeaway: flexible 'nonparametric' learning has dimension dependent scaling laws.

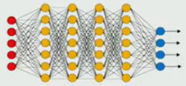
Source: CS-324, Stanford University

What if the loss-drop (i.e., emergence) follows power-law?

$y = ax^k$ → Power Law: *can be -ve* → $L(X) \propto 1/X^{\alpha_X}$

NO embeddin matrix.

$$L(N) \approx \left(\frac{N_c}{N}\right)^{\alpha_N}$$ $\alpha_N \sim 0.076$, $N_c \sim 8.8 \times 10^{13}$ (non-embedding parameters)



Model size (parameters)

→N

depends on tokenization.

Scale

$$L(D) \approx \left(\frac{D_c}{D}\right)^{\alpha_D}$$ $\alpha_D \sim 0.095$, $D_c \sim 5.4 \times 10^{13}$ (tokens)

Amount of training data

Kaplan Laws

$$L(C) \approx \left(\frac{C_c}{\boxed{C}}\right)^{\alpha_C}$$ $\alpha_C = 0.057$ $C_c = 1.6 \times 10^7$ PF-days

Amount of compute (or time)

$$L(C_{\min}) = \left(C_c^{\min}/\boxed{C_{\min}}\right)^{\alpha_C^{\min}}$$ $\alpha_C^{\min} \sim 0.050$, $C_c^{\min} \sim 3.1 \times 10^8$ (PF-days)

A more reliable version:

$$L(C_{\min}) = \left(C_c^{\min}/C_{\min}\right)^{\alpha_C^{\min}}$$

$$\alpha_C^{\min} \sim 0.050, \quad C_c^{\min} \sim 3.1 \times 10^8 \text{ (PF-days)}$$

$$C_{\min}(C) \equiv \frac{C}{1 + B/B_{\text{crit}}(L)} \quad \text{(minimum compute, at } B \ll B_{\text{crit}})$$

$$B_{\text{crit}}(L) \approx \frac{B_*}{L^{1/\alpha_B}}$$

$$B_* \sim 2 \cdot 10^8 \text{ tokens}, \quad \alpha_B \sim 0.21$$

Batch min. achieve.

$$L(C) \approx \left(\frac{C_c}{C}\right)^{\alpha_C}$$

Parameters	Data	Compute	Batch Size
Optimal	∞	C	Fixed

vs.

convert

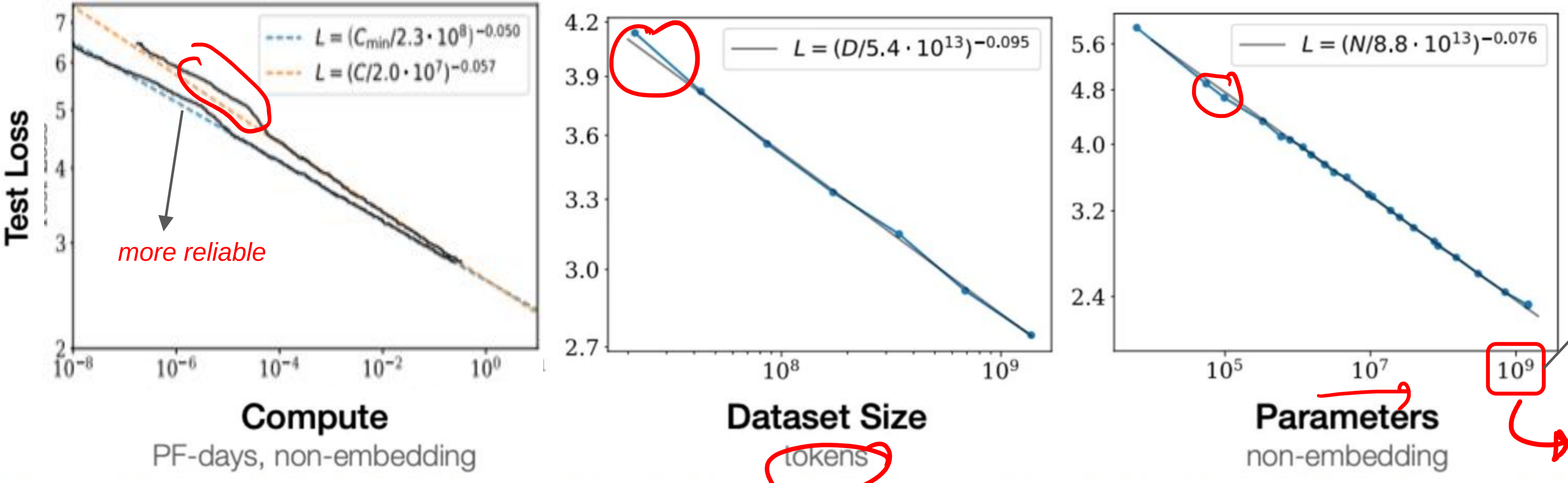
N_{opt}	D_{opt}	$C_{\min}$	$B \ll B_{\text{crit}}$

AA. LOT less.

Compute-Efficient Value	Power Law	Scale
$N_{\text{opt}} = N_e \cdot C_{\min}^{p_N}$	$p_N = 0.73$	$N_e = 1.3 \cdot 10^9$ params
$B \ll B_{\text{crit}} = \frac{B_*}{L^{1/\alpha_B}} = B_e C_{\min}^{p_B}$	$p_B = 0.24$	$B_e = 2.0 \cdot 10^6$ tokens

Kaplan Laws

We are in luck! Turns out that scale ***is*** predictable



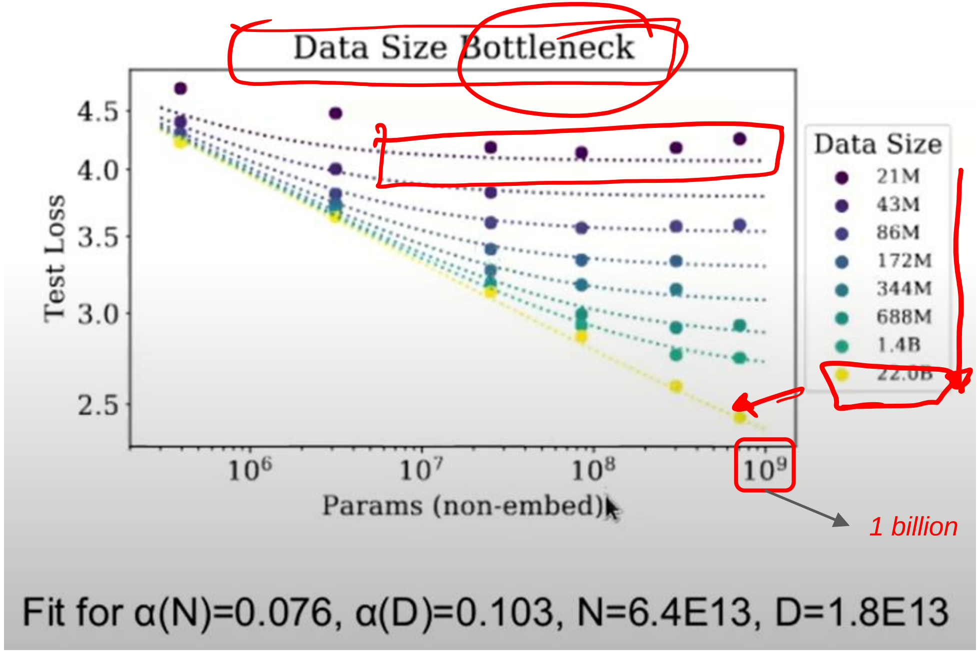
1 billion

vs.

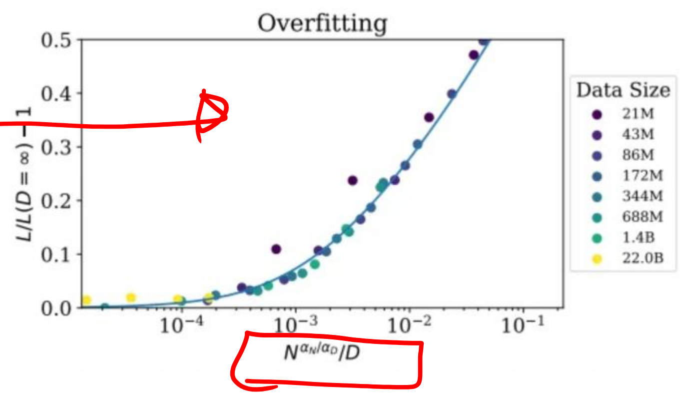
100 billion

(NT: log-log plotting) → Now that's smooth!!

Observation 1a: *Universality of Overfitting*

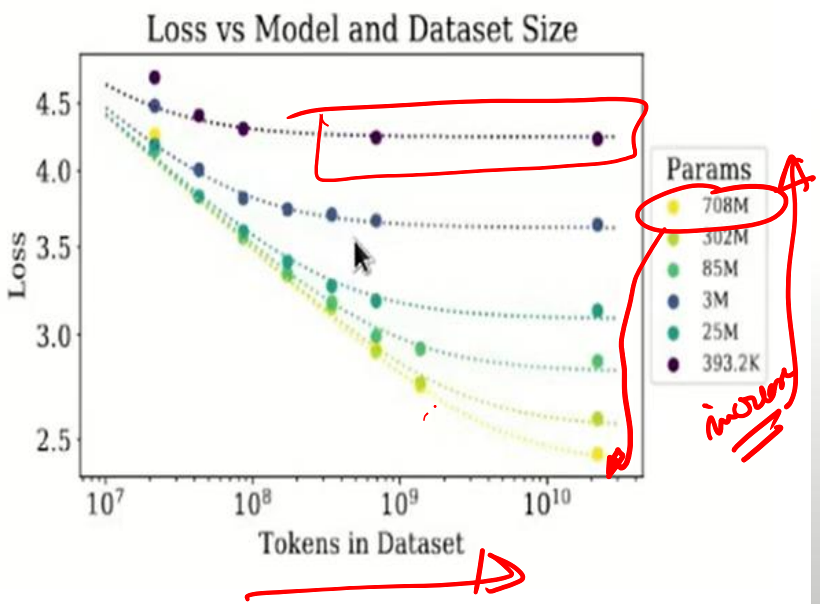


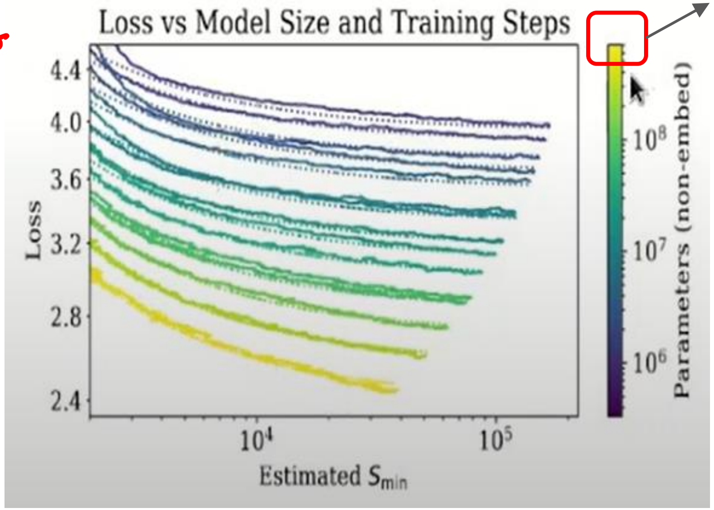
1 billion



2020 Arxiv (cs); Johns Hopkins University OpenAI

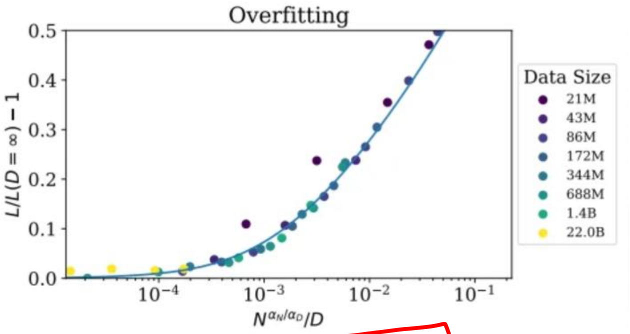
Observation 1b: *Sample Efficiency*





1 billion

Key takeaway 1: *Both* parameter and dataset to be scaled



$$D \propto N^{\frac{\alpha_N}{\alpha_D}} \sim N^{0.74}$$

Performance penalty is $N^{0.75}$ / D

- if model increases 8x, dataset must increase 5x

Both needs to be scaled together

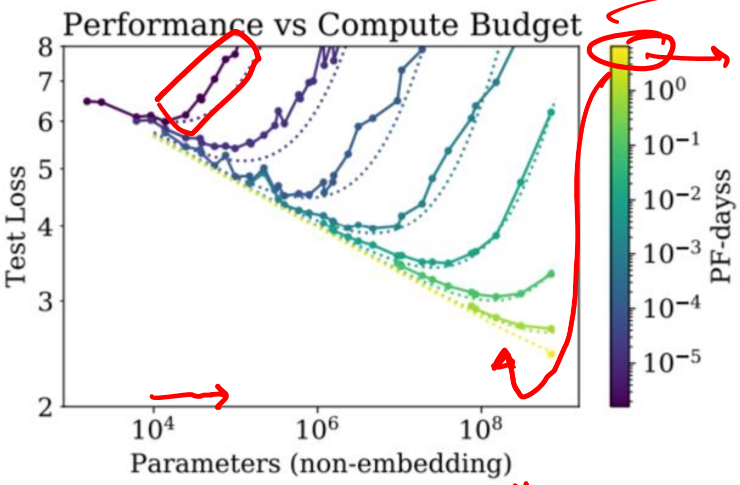
$$L(N) \approx \left(\frac{N_c}{N}\right)^{\alpha_N}$$

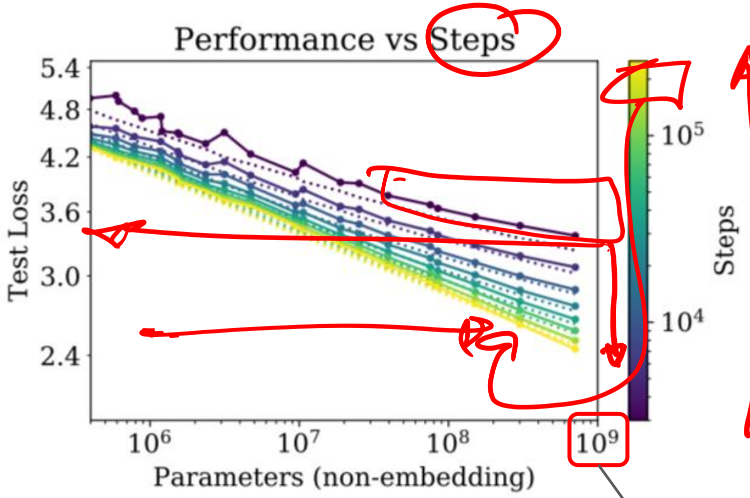
$$L(D) \approx \left(\frac{D_c}{D}\right)^{\alpha_D}$$

$$L(N,D) = \left[\left(\frac{N_c}{N}\right)^{\frac{\alpha_N}{\alpha_D}} + \frac{D_c}{D}\right]^{\alpha_D}$$

Joint scaling law.

Observation 2: What about training time (*steps & FLOPs*)?





1 billion

Key takeaway 2: *Universality of training*

not enough

$$L(N, S) = \left(\frac{N_c}{N}\right)^{\alpha_N} + \left(\frac{S_c}{S_{\min}(S)}\right)^{\alpha_S}$$

$$L(N, D) = \left[\left(\frac{N_c}{N}\right)^{\frac{\alpha_N}{\alpha_D}} + \frac{D_c}{D}\right]^{\alpha_D}$$

$S_c \approx 2.1 \times 10^3$ and $\alpha_S \approx 0.76$, and $S_{\min}(S)$ is the minimum possible number of optimization steps

$$S_{\min}(S) \equiv \frac{S}{1 + B_{\text{crit}}(L)/B} \qquad \text{(minimum steps, at } B \gg B_{\text{crit}}\text{)}$$

$$B_{\text{crit}}(L) \approx \frac{B_*}{L^{1/\alpha_B}}$$

$B_* \sim 2 \cdot 10^8$ tokens, $\alpha_B \sim 0.21$

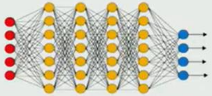
Parameter	α_N	α_D	N_c	D_c
Value	0.076	0.103	6.4×10^{13}	1.8×10^{13}

Parameter	α_N	α_S	N_c	S_c
Value	0.077	0.76	6.5×10^{13}	2.1×10^3

Kaplan Laws

Are we only to worry about

Amount of training data

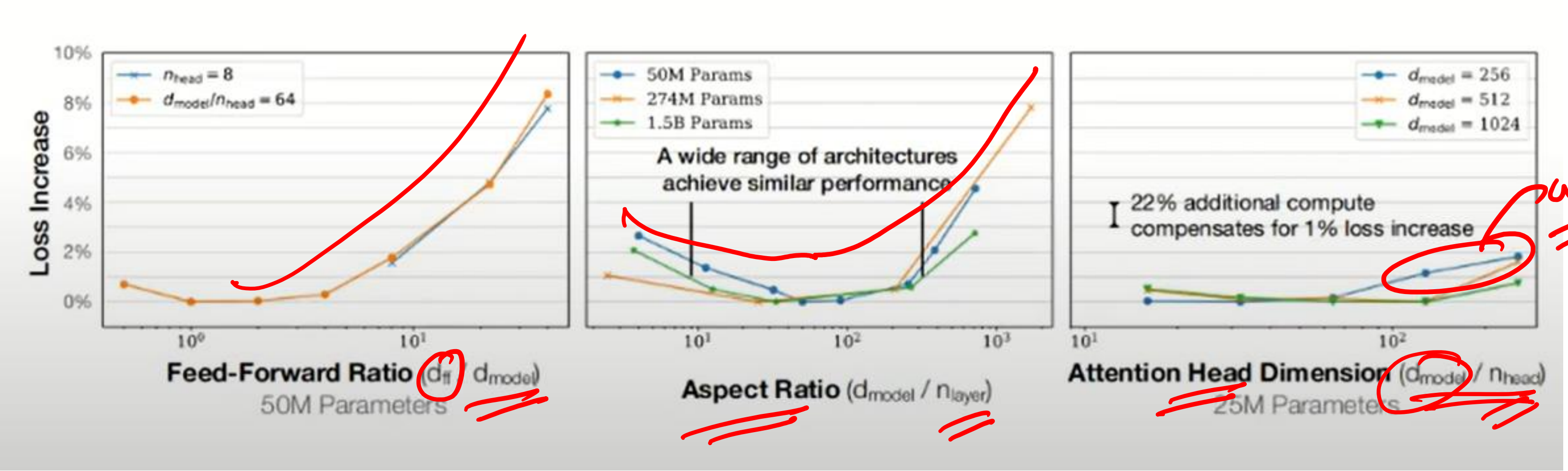


Model size (parameters)

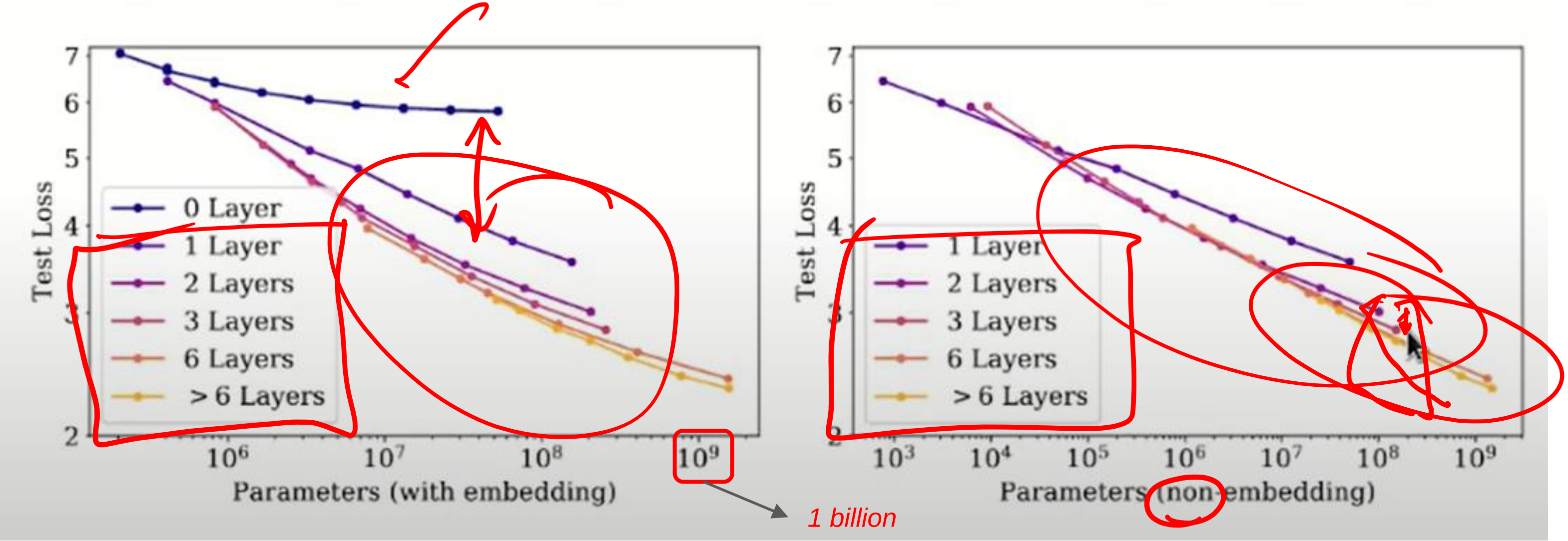
Amount of compute (or time)

???

Key Takeaway 3: Model shape does not matter!

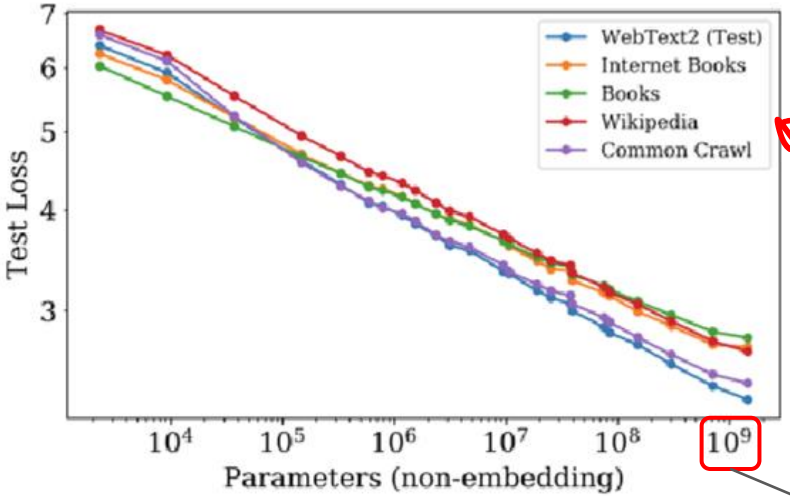


Key Takeaway 4: Embedding matrix does not matter!



Key Takeaway 5: Dataset *composition* does not matter!

Type of dataset.



1 billion

Kaplan Scaling Laws at a glance:

Power Law	Scale (tokenization-dependent)
$\alpha_N = 0.076$	$N_c = 8.8 \times 10^{13}$ params (non-embed)
$\alpha_D = 0.095$	$D_c = 5.4 \times 10^{13}$ tokens
$\alpha_C = 0.057$	$C_c = 1.6 \times 10^{7}$ PF-days
$\alpha_C^{\min} = 0.050$	$C_c^{\min} = 3.1 \times 10^{8}$ PF-days
$\alpha_B = 0.21$	$B_* = 2.1 \times 10^{8}$ tokens
$\alpha_S = 0.76$	$S_c = 2.1 \times 10^{3}$ steps

Parameters	Data	Compute	Batch Size	Equation
N	∞	∞	Fixed	$L(N) = (N_c/N)^{\alpha_N}$
∞	D	Early Stop	Fixed	$L(D) = (D_c/D)^{\alpha_D}$
Optimal	∞	C	Fixed	$L(C) = (C_c/C)^{\alpha_C}$ (naive)
N_{opt}	D_{opt}	$C_{\min}$	$B \ll B_{\text{crit}}$	$L(C_{\min}) = \left(C_c^{\min}/C_{\min}\right)^{\alpha_C^{\min}}$
N	D	Early Stop	Fixed	$L(N, D) = \left[\left(\frac{N_c}{N}\right)^{\frac{\alpha_N}{\alpha_D}} + \frac{D_c}{D}\right]^{\alpha_D}$
N	∞	S steps	B	$L(N, S) = \left(\frac{N_c}{N}\right)^{\alpha_N} + \left(\frac{S_c}{S_{\min}(S,B)}\right)^{\alpha_S}$

Joint.

Is there any other alternative law?

Turns out there is!

Total parameters.

Lower bound

$$\text{Loss}(N_T, D) = \frac{N_c}{N_T^{\alpha}} + \frac{D_c}{D^{\beta}} + E,$$

$$\text{Loss}(N_T, C_T) = \frac{N_c}{N_T^{\alpha}} + \frac{D_c}{(C_T/6N_T)^{\beta}} + E$$

Loss that is intrinsic to the dataset.

Chinchilla (Hoffman) Scaling Law

The Chinchilla (Hoffman) Scaling Law

$$\text{Loss}(N_T, D) = \frac{N_c}{N_T^{\alpha}} + \frac{D_c}{D^{\beta}} + E$$

$$L(N, D) = 1.69 + \frac{406.4}{N^{0.34}} + \frac{410.7}{D^{0.28}}$$

$$N_{opt}(C) = G(C/6)^{a} \quad D_{opt}(C) = G^{-1}(C/6)^{b}$$

$$\text{where} \quad G = \left(\frac{\alpha A}{\beta B}\right)^{\frac{1}{\alpha+\beta}} \quad a = \frac{\beta}{\alpha+\beta} \quad b = \frac{\alpha}{\alpha+\beta}$$

Fitting the constants, yields: $\alpha \approx \beta$

i.e. equal scaling of **N** and **D**.

best curve fitting

Chinchilla Scaling Law vs. Kaplan Scaling Law

$$\text{Kaplan:} N^*_{\setminus E} \propto C^{0.73}_{\setminus E}$$

$$\text{Chinchilla:} N^*_T \propto C^{0.50}_T.$$

without Embed Matrix

Performance penalty is $N^{0.75}$ / D

- if model increases 8x, dataset must increase 5x

vs.

Fitting the constants, yields: $\alpha \approx \beta$

i.e. equal scaling of ***N*** and ***D***.

params for Embed Matrix .

$$N_T = \boxed{N_E} + N_{\setminus E}, \qquad C_T = 6N_T D = 6(N_E + N_{\setminus E})D,$$

$$N_E = (h + v)d, \qquad C_{\setminus E} = 6N_{\setminus E} D.$$

context , vocab .

The (revised) Chinchilla Scaling Law

Scaling factor of Dataset.

$$L(N, D) = 1.82 + \frac{514.0}{N^{0.35}} + \frac{2115.2}{D^{0.37}}$$

better fit.

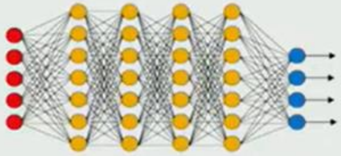
$$\boxed{L(N, D) = 1.69 + \frac{406.4}{N^{0.34}} + \frac{410.7}{D^{0.28}}}$$

→ Chinchilla law.

Is it a problem with our point-of-*view*?

LLMs "*seems*" to get more intelligent with the following:

Amount of training data

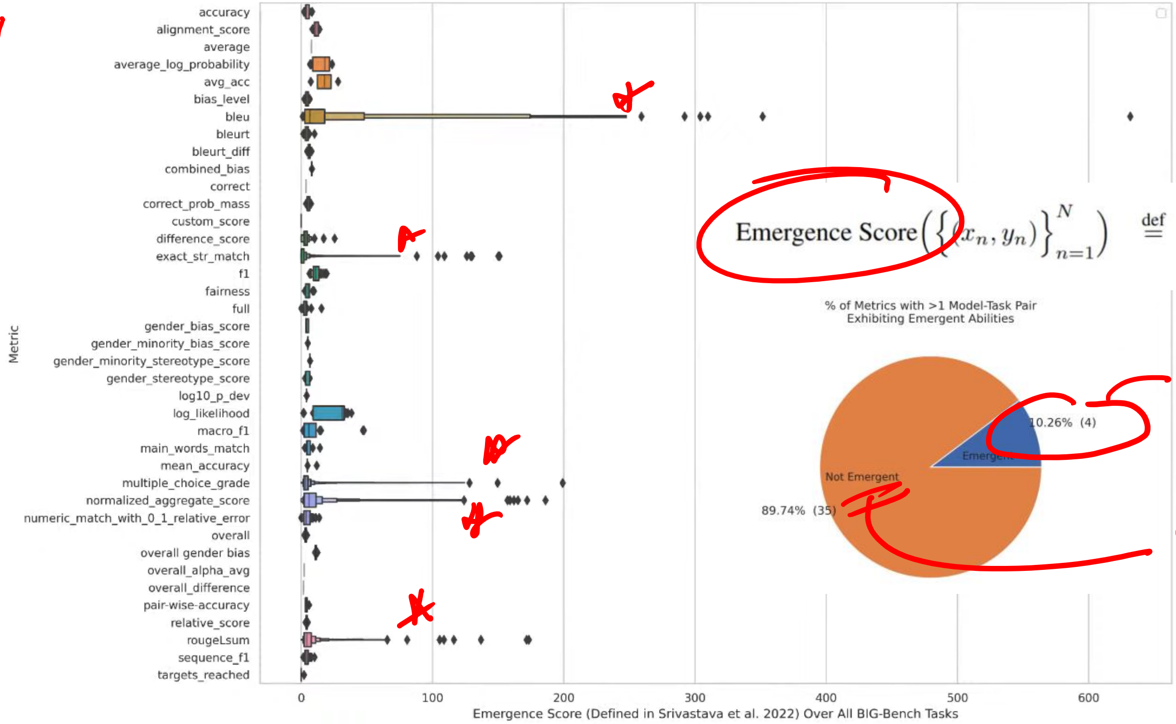


Model size (parameters)

$$\mathcal{L}_{CE}(N) = \left(\frac{N}{c}\right)^{\alpha}$$

Amount of compute (or time)

Motivation: Not all metrics score same (Emergence Score)



$$\text{Emergence Score}\left(\left\{(x_n, y_n)\right\}_{n=1}^{N}\right) \overset{\text{def}}{=} \frac{\text{sign}(\arg\max_i y_i - \arg\min_i y_i)(\max_i y_i - \min_i y_i)}{\sqrt{\text{Median}(\{(y_i - y_{i-1})^2\}_i)}}$$

10%

most measures

Is your accuracy metric non-linear or discontinuous?

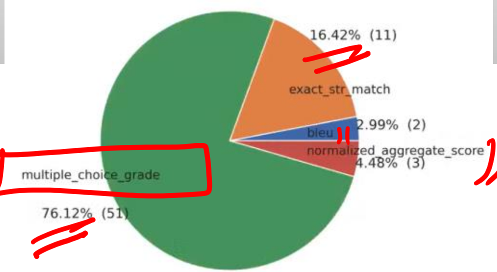
benchmark.

> 92% of BIG-BENCH:

smaller

$$\text{Multiple Choice Grade} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if highest probability mass on correct option} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Exact String Match} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if output string exactly matches target string} \\ 0 & \text{otherwise} \end{cases}$$

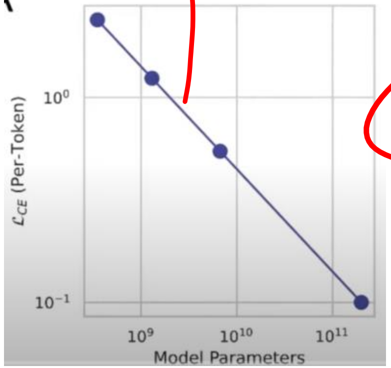


Too challenging for smaller models!
Is it really worth??

Power Law in play!

Kaplan.

Cross Entropy

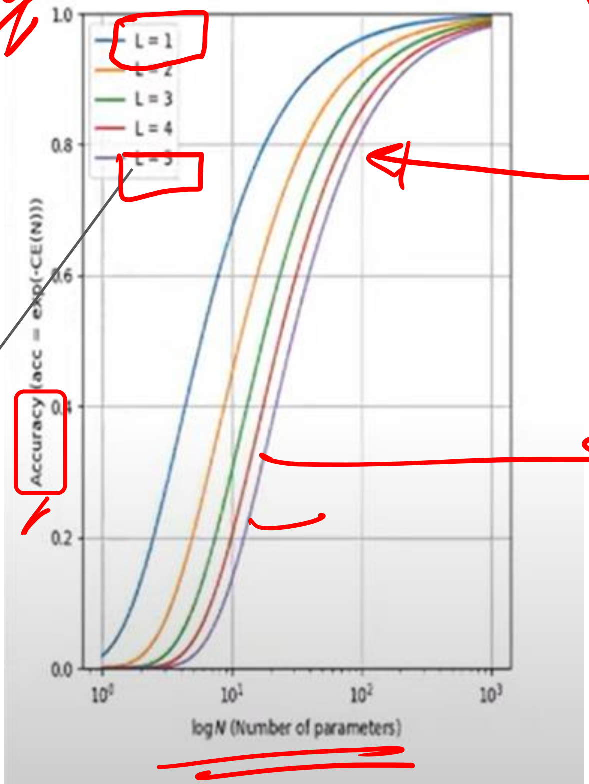


$$\mathcal{L}_{CE}(N) = \left(\frac{N}{c}\right)^{\alpha} = -\log \hat{p}_{v^*}(N)$$

$$\hat{p}_{v^*}(N) = \exp\left(-(N/c)^{\alpha}\right)$$

$$L(D_0, N) = B_0 + \frac{B_1}{N^{\alpha_W}} + \dots$$

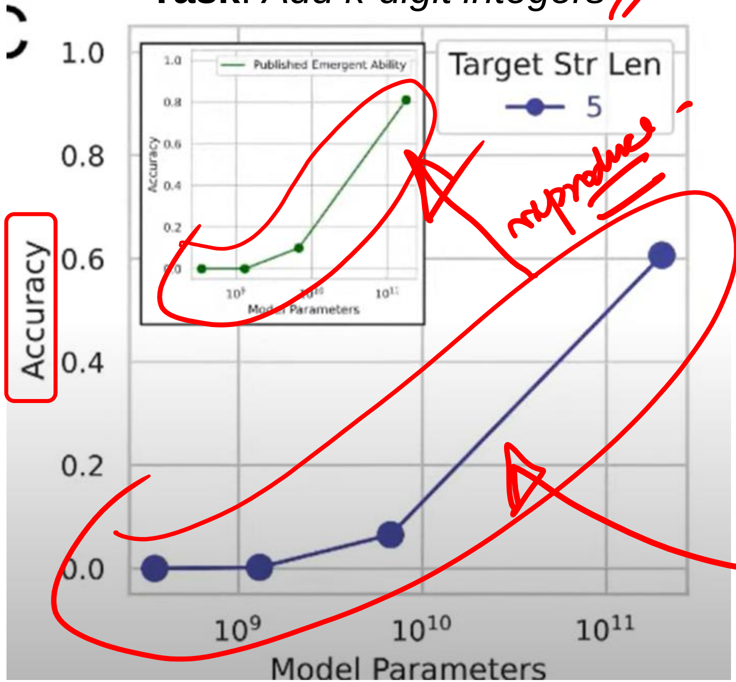
Length of token



S-curve.

Problem with Non-linear Measure: Eg.: *Exact string match*

Task: *Add k-digit integers*



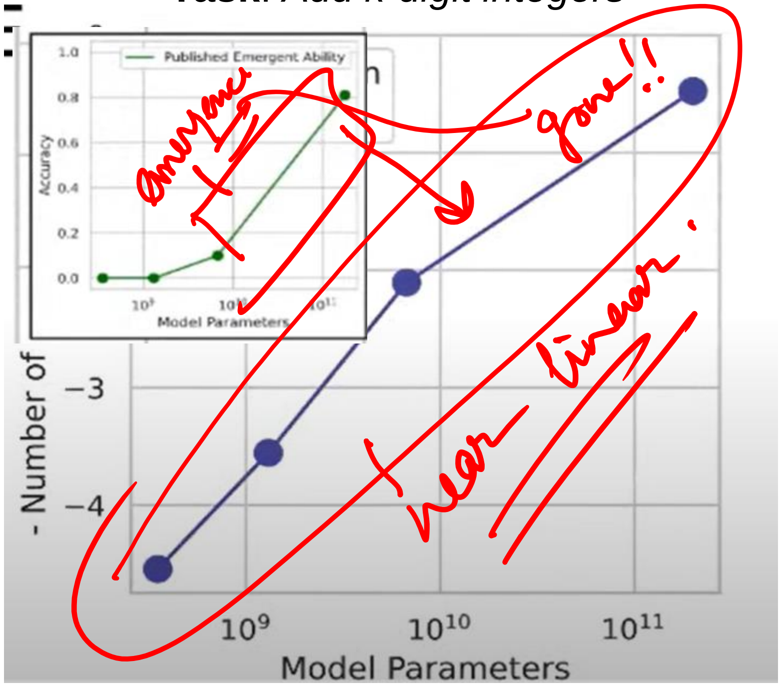
1	if all K+1 digits in model's output are correct
0	otherwise

$$\hat{p}_{v^*}(N) = \exp\left(-(N/c)^{\alpha}\right)$$

$$\text{Accuracy}(N) \approx p_N(\text{single token correct})^{\text{num. of tokens}}$$

Change of perspective: Measure: *Edit distance*

Task: *Add k-digit integers*



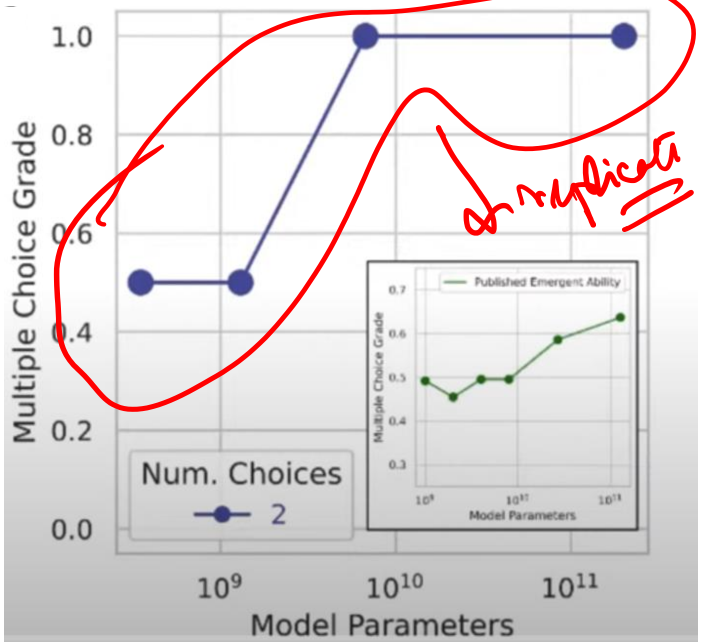
1	if all K+1 digits in model's output are correct
0	otherwise

$$\hat{p}_{v^*}(N) = \exp\left(-(N/c)^{\alpha}\right)$$

$$\text{Edit Distance}(N) \approx L\left(1 - p_N(\text{single token correct})\right)$$

Problem with Discontinuous Measure: Eg.: *MCG*

Task: *Choose one of two*

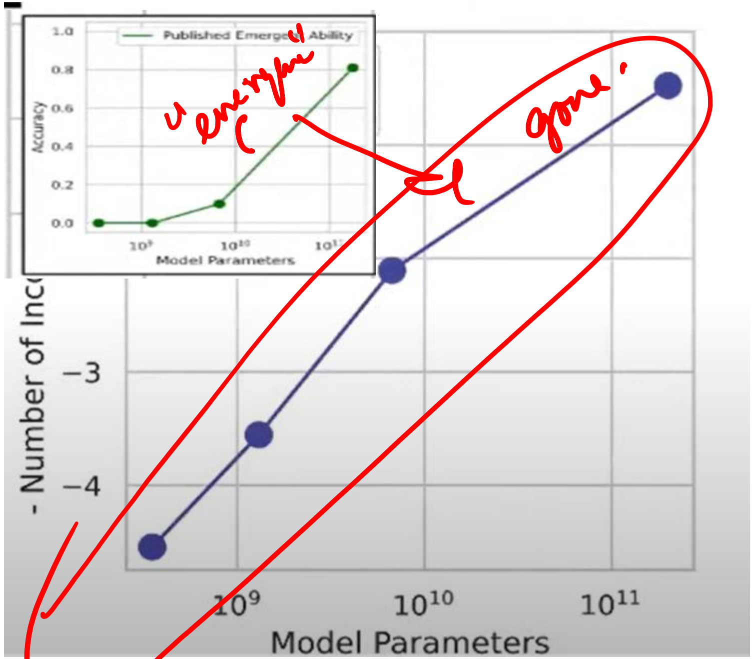


1	if highest probability mass on correct option
0	otherwise

$$\hat{p}_{v^*}(N) = \exp\left(- (N/c)^{\alpha} \right)$$

Change of perspective: Measure: *Brier Score*

Task: *Choose one of two*

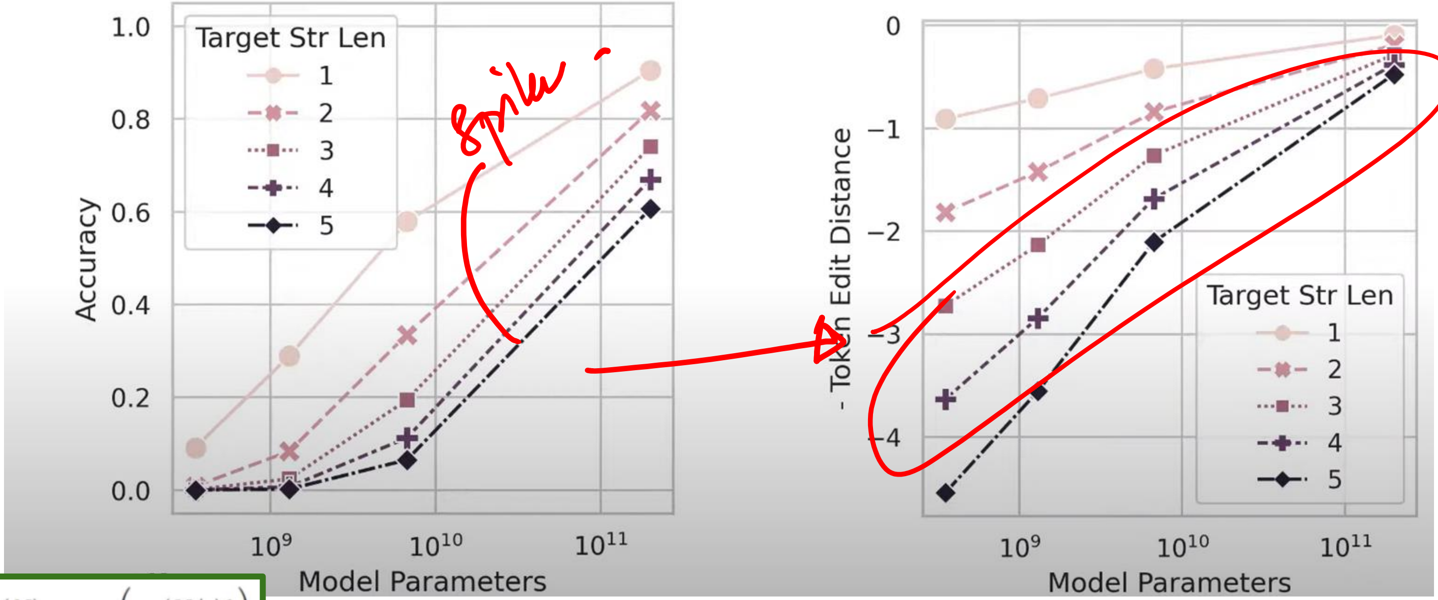


1	if all K+1 digits in model's output are correct
0	otherwise

$$\hat{p}_{v^*}(N) = \exp\left(-(N/c)^{\alpha}\right)$$

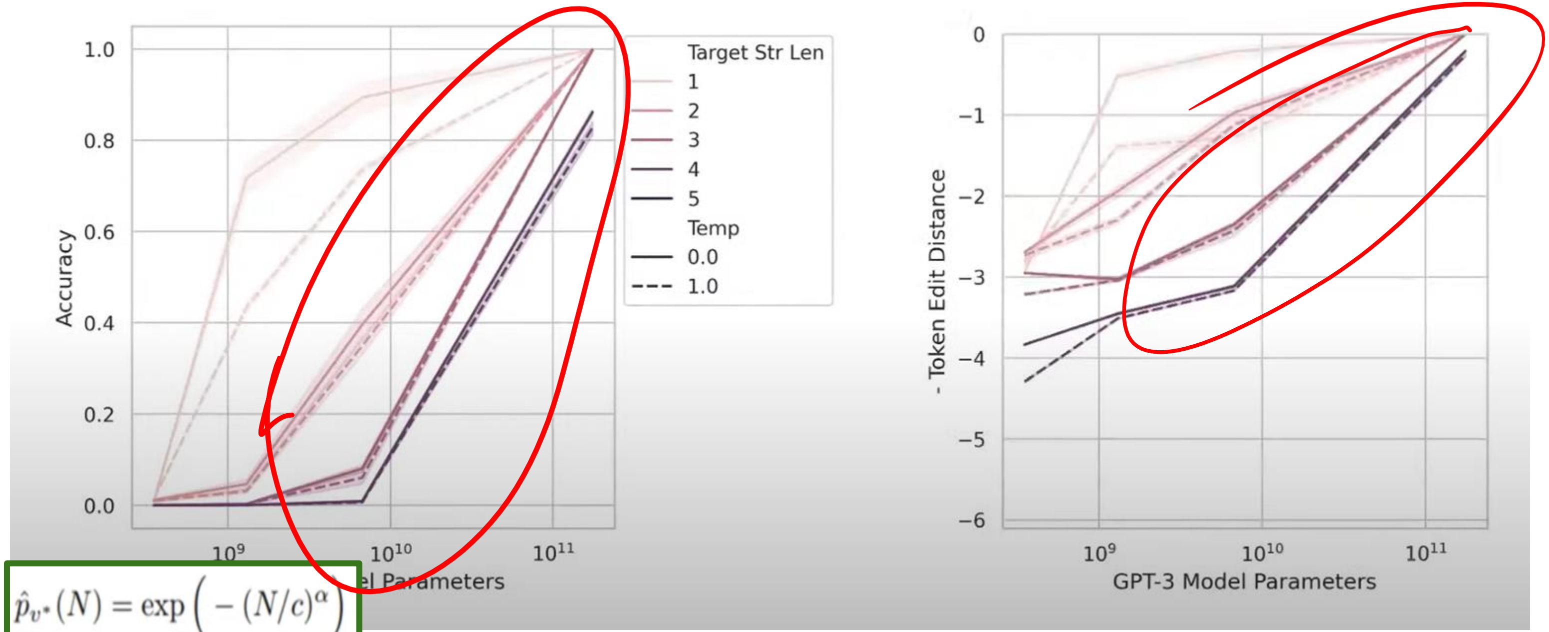
Brier Score = (1 – probability mass on correct option)^2

Prediction: Power Law vs. Near-Linear counterpart



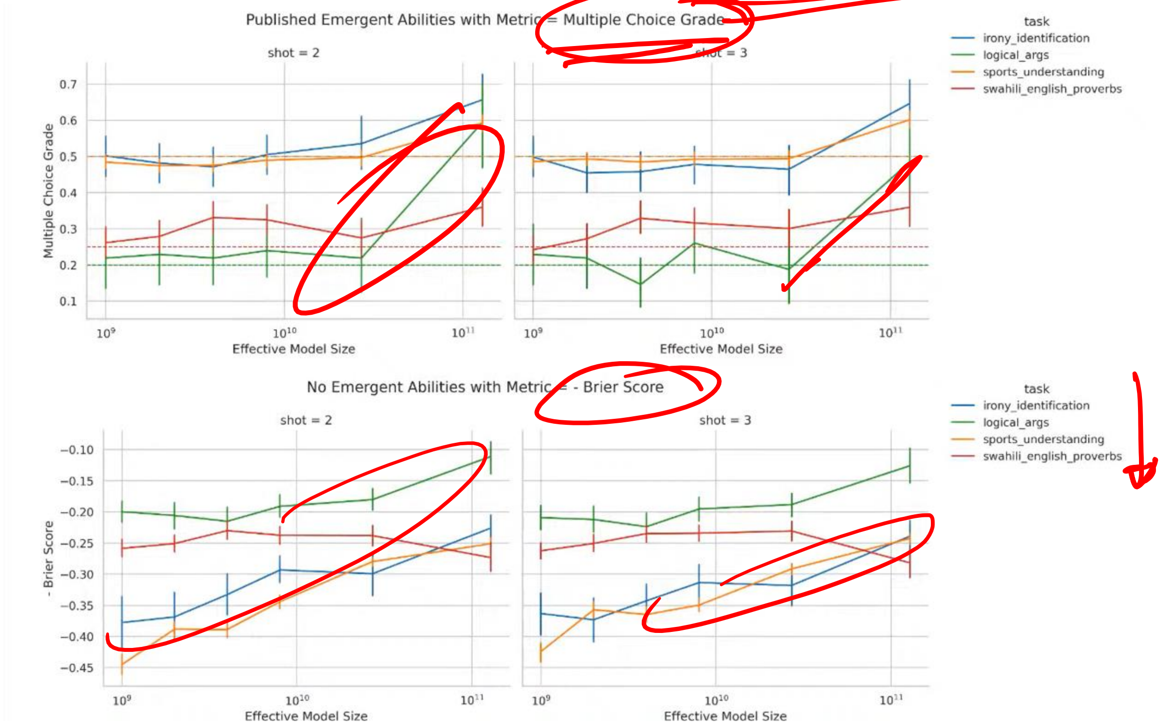
$$\hat{p}_{v^*}(N) = \exp\left(-(N/c)^{\alpha}\right)$$

Results on GPT3.5/3: Task: 2-digit integer multiplication



$$\hat{p}_{v^*}(N) = \exp\left(-(N/c)^{\alpha}\right)$$

Does the claim work for Google BIG-BENCH benchmark?



Key Takeaways

- Want to predict without the theatrics? Choose a *metric that's "soft"* (in the continuous sense)

- There's *no sudden jump* in reality ("*most*" can be predicted on a near-linear scale)

- Do we really need the power law of scale? Maybe not!