# Paul's Online Notes



## Section 14.5 : Lagrange Multipliers

In the previous section we optimized (*i.e.* found the absolute extrema) a function on a region that contained its boundary. Finding potential optimal points in the interior of the region isn't too bad in general, all that we needed to do was find the critical points and plug them into the function. However, as we saw in the examples finding potential optimal points on the boundary was often a fairly long and messy process.

In this section we are going to take a look at another way of optimizing a function subject to given constraint(s). The constraint(s) may be the equation(s) that describe the boundary of a region although in this section we won't concentrate on those types of problems since this method just requires a general constraint and doesn't really care where the constraint came from.

So, let's get things set up. We want to optimize (*i.e.* find the minimum and maximum value of) a function, $f\left(x,y,z\right)$, subject to the constraint $g\left(x,y,z\right)=k$. Again, the constraint may be the equation that describes the boundary of a region or it may not be. The process is actually fairly simple, although the work can still be a little overwhelming at times.

#### Method of Lagrange Multipliers

1. Solve the following system of equations.

$$\begin{aligned}\nabla f\left(x,y,z\right)&=\lambda\ \nabla g\left(x,y,z\right)\\ g\left(x,y,z\right)&=k\end{aligned}$$

2. Plug in all solutions, $\left(x,y,z\right)$, from the first step into $f\left(x,y,z\right)$ and identify the minimum and maximum values, provided they exist and $\nabla g\neq\vec{0}$ at the point.

The constant, $\lambda$, is called the **Lagrange Multiplier**.

Notice that the system of equations from the method actually has four equations, we just wrote the system in a simpler form. To see this let's take the first equation and put in the definition of the gradient vector to see what we get.

$$\left\langle f_x, f_y, f_z \right\rangle = \lambda \left\langle g_x, g_y, g_z \right\rangle = \left\langle \lambda g_x, \lambda g_y, \lambda g_z \right\rangle$$

In order for these two vectors to be equal the individual components must also be equal. So, we actually have three equations here.

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$$

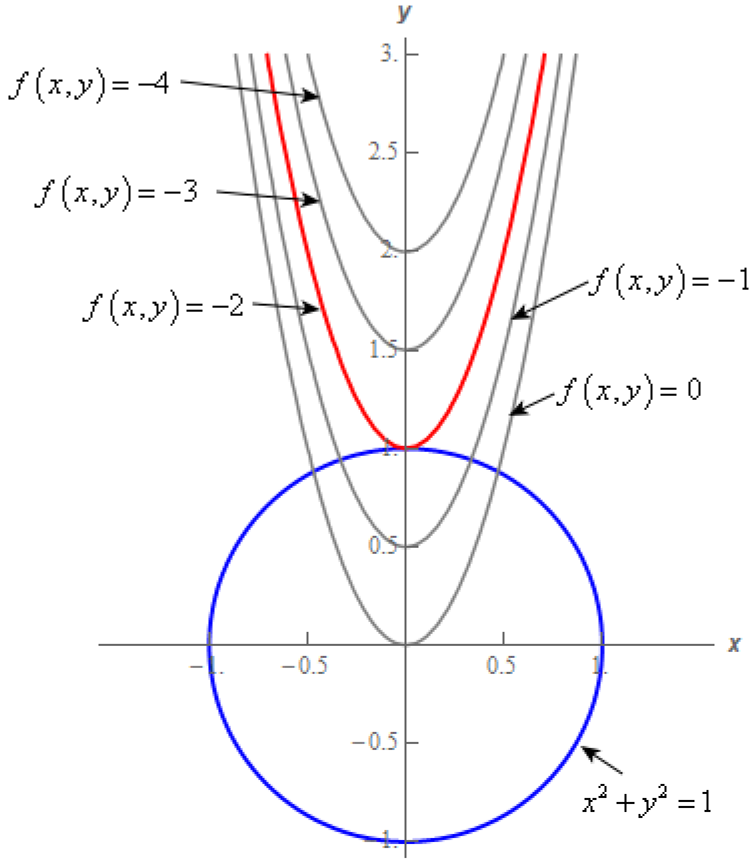
These three equations along with the constraint, $g\left(x, y, z\right) = c$, give four equations with four unknowns $x$, $y$, $z$, and $\lambda$.

Note as well that if we only have functions of two variables then we won't have the third component of the gradient and so will only have three equations in three unknowns $x$, $y$, and $\lambda$.

As a final note we also need to be careful with the fact that in some cases minimums and maximums won't exist even though the method will seem to imply that they do. In every problem we'll need to make sure that minimums and maximums will exist before we start the problem.

To see a physical justification for the formulas above let's consider the minimum and maximum value of $f\left(x, y\right) = 8x^2 - 2y$ subject to the constraint $x^2 + y^2 = 1$. In the practice problems for this section (problem #2 to be exact) we will show that minimum value of $f\left(x, y\right)$ is -2 which occurs at $\left(0, 1\right)$ and the maximum value of $f\left(x, y\right)$ is 8.125 which occurs at $\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$ and $\left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$.

Here is a sketch of the constraint as well as $f\left(x. y\right) = k$ for various values of $k$.




First remember that solutions to the system must be somewhere on the graph of the constraint, $x^2 + y^2 = 1$ in this case. Because we are looking for the minimum/maximum value of $f(x, y)$ this, in turn, means that the location of the minimum/maximum value of $f(x, y)$, *i.e.* the point $(x, y)$, must occur where the graph of $f(x, y) = k$ intersects the graph of the constraint when $k$ is either the minimum or maximum value of $f(x, y)$.

Now, we can see that the graph of $f(x, y) = -2$, *i.e.* the graph of the minimum value of $f(x, y)$, just touches the graph of the constraint at $(0, 1)$. In fact, the two graphs at that point are tangent.

If the two graphs are tangent at that point then their normal vectors must be parallel, *i.e.* the two normal vectors must be scalar multiples of each other. Mathematically, this means,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

for some scalar $\lambda$ and this is exactly the first equation in the system we need to solve in the method.

Note as well that if $k$ is smaller than the minimum value of $f(x, y)$ the graph of $f(x, y) = k$ doesn't intersect the graph of the constraint and so it is not possible for the function to take that value of $k$ at a point that will satisfy the constraint.

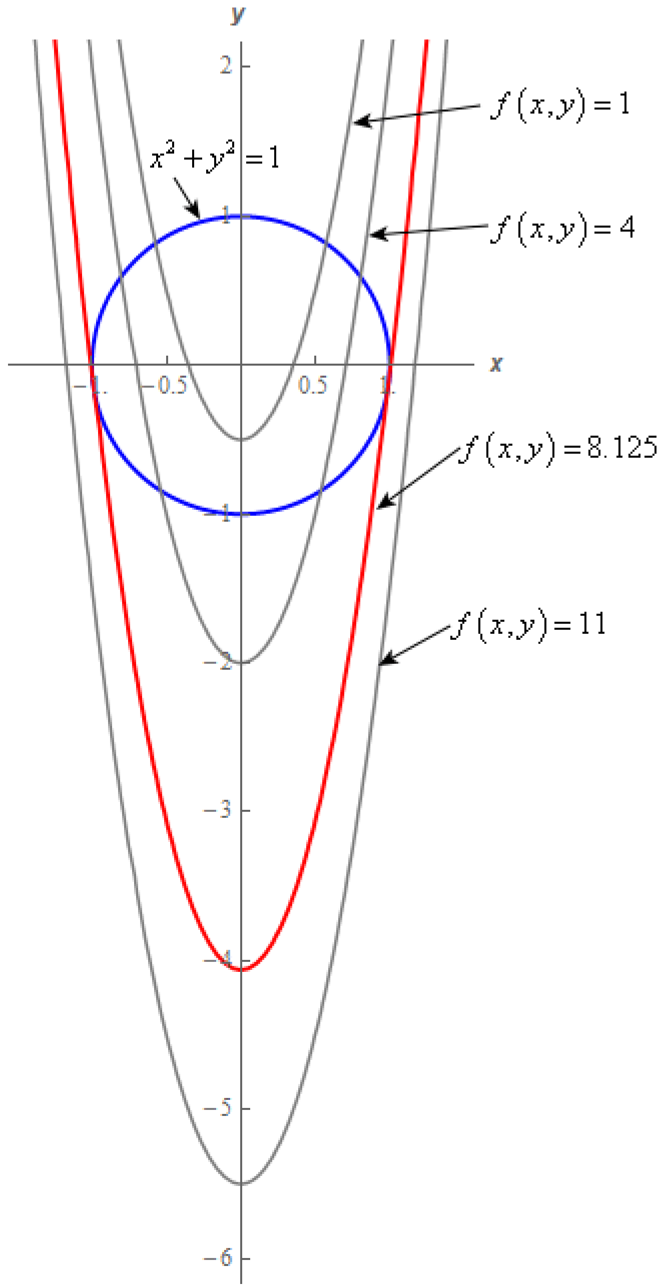
Likewise, if $k$ is larger than the minimum value of $f\left(x,y\right)$ the graph of $f\left(x,y\right)=k$ will intersect the graph of the constraint but the two graphs are not tangent at the intersection point(s). This means that the method will not find those intersection points as we solve the system of equations.

Next, the graph below shows a different set of values of $k$. In this case, the values of $k$ include the maximum value of $f\left(x,y\right)$ as well as a few values on either side of the maximum value.

Again, we can see that the graph of $f\left(x,y\right)=8.125$ will just touch the graph of the constraint at two points. This is a good thing as we know the solution does say that it should occur at two points. Also note that at those points again the graph of $f\left(x,y\right)=8.125$and the constraint are tangent and so, just as with the minimum values, the normal vectors must be parallel at these points.

Likewise, for value of $k$ greater than 8.125 the graph of $f\left(x,y\right)=k$ does not intersect the graph of the constraint and so it will not be possible for $f\left(x,y\right)$ to take on those larger values at points that are on the constraint.

Also, for values of $k$ less than 8.125 the graph of $f\left(x,y\right)=k$ does intersect the graph of the constraint but will not be tangent at the intersection points and so again the method will not produce these intersection points as we solve the system of equations.




So, with these graphs we've seen that the minimum/maximum values of $f(x,y)$ will come where the graph of $f(x,y)=k$ and the graph of the constraint are tangent and so their normal vectors are parallel. Also, because the point must occur on the constraint itself. In other words, the system of equations we need to solve to determine the minimum/maximum value of $f(x,y)$ are exactly those given in the above when we introduced the method.

Note that the physical justification above was done for a two dimensional system but the same justification can be done in higher dimensions. The difference is that in higher dimensions we won't be working with curves. For example, in three dimensions we would be working with surfaces. However, the same ideas will still hold. At the points that give minimum and maximum value(s) of the surfaces would be parallel and so the normal vectors would also be parallel.

Let's work a couple of examples.

***Example 1*** Find the dimensions of the box with largest volume if the total surface area is 64 $\text{cm}^2$.

***Show Solution***

Notice that we never actually found values for $\lambda$ in the above example. This is fairly standard for these kinds of problems. The value of $\lambda$ isn't really important to determining if the point is a maximum or a minimum so often we will not bother with finding a value for it. On occasion we will need its value to help solve the system, but even in those cases we won't use it past finding the point.

***Example 2*** Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

***Show Solution***

In the first two examples we've excluded $\lambda = 0$ either for physical reasons or because it wouldn't solve one or more of the equations. Do not always expect this to happen. Sometimes we will be able to automatically exclude a value of $\lambda$ and sometimes we won't.

Let's take a look at another example.

***Example 3*** Find the maximum and minimum values of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

***Show Solution***

Before we proceed we need to address a quick issue that the last example illustrates about the method of Lagrange Multipliers. We found the absolute minimum and maximum to the function. However, what we did not find is all the locations for the absolute minimum. For example, assuming $x, y, z \geq 0$, consider the following sets of points.

$$\begin{array}{ll} (0, y, z) & \text{where} \quad y + z = 1 \\ (x, 0, z) & \text{where} \quad x + z = 1 \\ (x, y, 0) & \text{where} \quad x + y = 1 \end{array}$$

Every point in this set of points will satisfy the constraint from the problem and in every case the function will evaluate to zero and so also give the absolute minimum.

So, what is going on? Recall from the previous section that we had to check both the critical points and the boundaries to make sure we had the absolute extrema. The same was true in Calculus I. We had to check both critical points and end points of the interval to make sure we had the absolute extrema.

It turns out that we really need to do the same thing here if we want to know that we've found all the locations of the absolute extrema. The method of Lagrange multipliers will find the absolute extrema, it just might not find all the locations of them as the method does not take the end points of variables ranges into account (note that we might luck into some of these points but we can't guarantee that).

So, after going through the Lagrange Multiplier method we should then ask what happens at the end points of our variable ranges. For the example that means looking at what happens if $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, and $z = 1$. In the first three cases we get the points listed above that do happen to also give the absolute minimum. For the later three cases we can see that if one of the variables are 1 the other two must be zero (to meet the constraint) and those were actually found in the example. Sometimes that will happen and sometimes it won't.

In the case of this example the end points of each of the variable ranges gave absolute extrema but there is no reason to expect that to happen every time. In Example 2 above, for example, the end points of the ranges for the variables do not give absolute extrema (we'll let you verify this).

The moral of this is that if we want to know that we have every location of the absolute extrema for a particular problem we should also check the end points of any variable ranges that we might have. If all we are interested in is the value of the absolute extrema then there is no reason to do this.

Okay, it's time to move on to a slightly different topic. To this point we've only looked at constraints that were equations. We can also have constraints that are inequalities. The process for these types of problems is nearly identical to what we've been doing in this section to this point. The main difference between the two types of problems is that we will also need to find all the critical points that satisfy the inequality in the constraint and check these in the function when we check the values we found using Lagrange Multipliers.

Let's work an example to see how these kinds of problems work.

***Example 4*** Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

***Show Solution*** ▸

The final topic that we need to discuss in this section is what to do if we have more than one constraint. We will look only at two constraints, but we can naturally extend the work here to more than two constraints.

We want to optimize $f(x, y, z)$ subject to the constraints $g(x, y, z) = c$ and $h(x, y, z) = k$. The system that we need to solve in this case is,

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) &= c \\ h(x, y, z) &= k \end{aligned}$$

So, in this case we get two Lagrange Multipliers. Also, note that the first equation really is three equations as we saw in the previous examples. Let's see an example of this kind of optimization problem.

***Example 5*** Find the maximum and minimum of $f(x, y, z) = 4y - 2z$ subject to the constraints $2x - y - z = 2$ and $x^2 + y^2 = 1$.

***Show Solution*** ▸

 Page Last Modified : 4/17/2023