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Generalized Linear Models (GLM)

Linear Regression / 1807

1. $y_i \leftarrow$ independent
2. y_i = dependent variable = random variable of prediction

   $\sim N(\mu_i, \sigma^2)$

3. $\underline{\mu_i} = \underline{\theta^T x_i} = \theta \cdot x_i^{1/2} = \theta \cdot \sin(x_i)$ $\leftarrow$ predictor function
4. $\theta \Rightarrow$ LSE $\leftarrow$

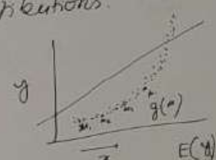
   MLE $\leftarrow$

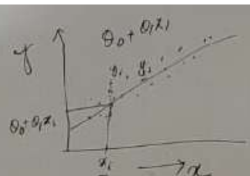
~1960

1.
2. $y_i \sim$ exponential family of distributions.
3. $g(\mu) = \theta^T x$ $\leftarrow$ link fn

   $g(\cdot) = \log(\mu)$ $\quad g(x) = x$

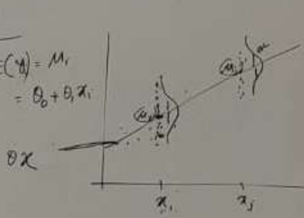
   $g(\cdot) =$
4. MLE





$E(y) = \mu_i$

$= \theta_0 + \theta_1 x_i$



$$f_y(y|\theta,\varphi) = \exp\left(\frac{y\cdot\theta - b(\theta)}{a(\varphi)} + c(y,\varphi)\right) \to \text{exponential family}$$

Normal

$$f_y(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(\log\frac{1}{\sqrt{2\pi\sigma^2}} + \frac{-(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{1}{2}\log(2\pi\sigma^2) + \frac{-(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(-\frac{y^2}{2\sigma^2} + \frac{\mu y}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right)$$

$$= \exp\left(\frac{y\mu - \frac{1}{2}\mu^2}{\sigma^2} - \frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)\right)$$

$$= \exp\left(\frac{y\cdot\theta - \frac{1}{2}\theta^2}{\varphi} - \frac{1}{2}\left(\quad\right)\right)$$

Goal: θ, φ, $b(\cdot)$, $a(\cdot)$, $c(\cdot)$

$\boxed{\theta = \mu}$, $\varphi = \sigma^2$

$b(\theta) = \frac{1}{2}\theta^2$

$a(\varphi) = \varphi$

$$c(y,\varphi) = -\frac{1}{2}\left(\frac{y^2}{\varphi} + \log(2\pi\varphi)\right)$$

Poisson

$$f_Y(y \mid \theta, \phi) = \exp(-\mu) \frac{\mu^y}{y!}$$

$$= \exp(-\mu) \exp\left(\log\left(\frac{\mu^y}{y!}\right)\right)$$

$$= \exp(-\mu) \exp\left(\log(\mu^y) - \log(y!)\right)$$

$$= \exp\left(-\mu + \log(\mu^y) - \log(y!)\right)$$

$\theta = \log \mu$

$$= \exp\left(y\theta - \underline{\exp(\theta)} - \log(y!)\right)$$

$\phi = 1$

$a(\phi) = 1$

$b(\theta) = \exp(\theta)$

$c(y, \phi) = -\log(y!)$

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Generalized Linear Models (GLM)

Binomial

n : no of trials

$\pi = P_r(y=1)$

$$f_y(y|\theta,\varphi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$$

$$= \exp\left(\log\left[\binom{n}{y}\pi^y(1-\pi)^{n-y}\right]\right)$$

$$= \exp\left(\log\binom{n}{y} + \log\pi^y + \log(1-\pi)^{(n-y)}\right)$$

$$= \exp\left(\log\binom{n}{y} + y\log\pi + (n-y)\log(1-\pi)\right)$$

$$= \exp\left(\log\binom{n}{y} + y\log\pi + n\log(1-\pi) - y\log(1-\pi)\right)$$

$$= \exp\left(y\left(\log\pi - \log(1-\pi)\right) + n\log(1-\pi) + \log\binom{n}{y}\right)$$

$$= \exp\left(y\cdot\underbrace{\log\frac{\pi}{1-\pi}}_{\theta} + n\log(1-\pi) + \log\binom{n}{y}\right)$$

$a(\varphi)=1$

$\varphi=1$

$$\theta = \log\frac{\pi}{1-\pi}$$

$$\Rightarrow e^\theta = \frac{\pi}{1-\pi}$$

$$\Rightarrow \pi = \frac{e^\theta}{1+e^\theta}$$

$$\log(1-\pi) = \log\left(\frac{1}{1+e^\theta}\right)$$

$$= -\log(1+e^\theta)$$

$\boxed{\theta = \log\frac{\pi}{1-\pi}}$; $\varphi = 1$; $a(\varphi)=1$

$b(\theta) = n\log(1+\exp(\theta))$

$c(y,\varphi) = \log\binom{n}{y}$

Gamma

Inverse Gaussian

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Generalized Linear Models (GLM)

Prove: $E(y) = b'(\theta) = \mu$.

$$\ell(\theta, \phi) = \frac{y\,\theta - b(\theta)}{a(\phi)} + c(y, \phi)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{y - b'(\theta)}{a(\phi)} + 0$$

We know

$$E\left(\frac{\partial \ell}{\partial \theta}\right) = 0 \Longrightarrow E\left(\frac{\partial \ell}{\partial \theta}\right) = E\left(\frac{y - b'(\theta)}{a(\phi)}\right) = 0$$

$$\Downarrow$$

score fn

$\Rightarrow$ expectation of the log of the PDF of the prob. dist.

$$\Rightarrow \frac{E(y) - b'(\theta)}{a(\phi)} = 0$$

$$\boxed{\Rightarrow E(y) = b'(\theta)}$$

$Var(y) = b''(\theta) . a(\phi)$

Prove: $E\left(\frac{\partial \ell}{\partial \theta}\right) = 0$

$$\Rightarrow E\left(\frac{\partial}{\partial \theta} \log f_y(\cdot)\right)$$

$$= E\left(\frac{1}{f_y(\cdot)} \frac{\partial}{\partial \theta} f_y(\cdot)\right)$$

$$= \int \cancel{f_y(\cdot)} \frac{1}{\cancel{f_y(\cdot)}} \frac{\partial}{\partial \theta} f_y(\cdot)\, dy$$

$$= \int \frac{\partial}{\partial \theta} f_y(\cdot)\, dy$$

$$= \frac{\partial}{\partial \theta} \int f_y(\cdot)\, dy$$

$$= \frac{\partial}{\partial \theta} 1 = 0$$

Generalized Linear Models (GLM)

$$E\left(\frac{d^2 l}{\partial\theta^2}\right) = -E\left[\left(\frac{dl}{\partial\theta}\right)^2\right]$$

Log-Likelihood: $l = \frac{y\theta - b(\theta)}{a(\phi)} + c(\phi, y)$

$$\frac{\partial l}{\partial\theta} = \frac{y - b'(\theta)}{a(\phi)} \quad \text{--- (i)}$$

$$\Rightarrow \frac{\partial^2 l}{\partial\theta^2} = -\frac{b''(\theta)}{a(\phi)} \quad \text{--- (ii)}$$

$$\rightarrow E\left[-\frac{b''(\theta)}{a(\theta)}\right] = -E\left[\left(\frac{y - b'(\theta)}{a(\phi)}\right)^2\right]$$

$$\Rightarrow \cancel{-}\frac{b''(\theta)}{\cancel{a(\phi)}} = \cancel{-}E\left[\frac{(y-\mu)^2}{(a(\phi))^{\cancel{2}}}\right]$$

$$\Rightarrow E\left[(y-\mu)^2\right] = a(\theta)\cdot b''(\theta)$$

$$\boxed{Var(y) = b''(\theta).a(\phi)}$$

$$E\left(\frac{d^2 l}{\partial\theta^2}\right) = E\left(\frac{d^2}{\partial\theta^2}\log f_y(\cdot)\right) = E\left(\frac{\partial}{\partial\theta}\left(\frac{1}{f_y(\cdot)}\frac{\partial}{\partial\theta}f_y(\cdot)\right)\right)$$

$$= E\left(-\frac{1}{f_y^2(\cdot)}\frac{\partial}{\partial\theta}f_y(\cdot)\frac{\partial}{\partial\theta}f_y(\cdot) + \frac{1}{f_y(\cdot)}\frac{\partial^2}{\partial\theta^2}f_y(\cdot)\right)$$

$$= -E\left[\left(\frac{1}{f_y(\cdot)}\frac{\partial}{\partial\theta}f_y(\cdot)\right)^2\right] + E\left(\frac{1}{f_y(\cdot)}\frac{\partial^2}{\partial\theta^2}f_y(\cdot)\right)$$

0

$$\frac{\partial}{\partial\theta}\log f_y(\cdot) = \frac{1}{f_y(\cdot)}\frac{\partial}{\partial\theta}f_y(\cdot)$$

$$= -E\left(\left[\frac{\partial}{\partial\theta}\log f_y(\cdot)\right]^2\right)$$

$$= -E\left[\left(\frac{\partial L}{\partial\theta}\right)^2\right]$$

$$f_y(y|\theta,\phi) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right) \rightarrow \text{exponential family}$$

Link fn^n

We are linking y (random variable) with systematic component $x_1, \ldots, x_n$ via $g(\cdot)$

covariates produce linear predictor $\eta = W^T X$

Normal: $\theta = \mu = E(y) = W^T X \Rightarrow$ Identity link

Binomial $\theta = \log \frac{\pi}{1-\pi} \leftarrow$ logit link

poisson $\theta = \log \mu \leftarrow$ log link

Complementary log-log link

$$\eta = \log(-\log(1-\pi))$$