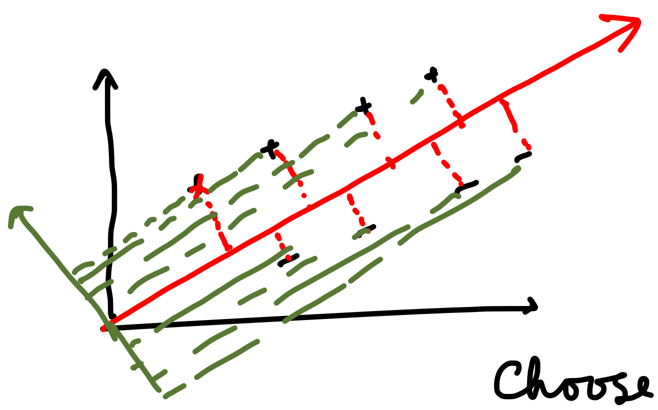
21/10/24

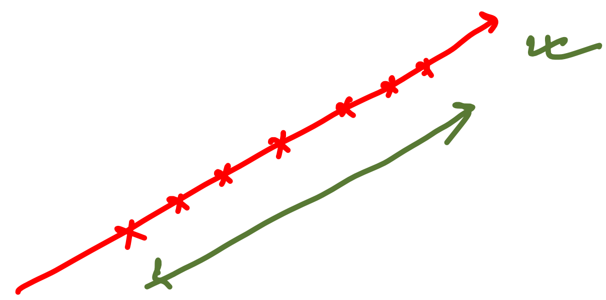
Dimensionality Reduction

PCA, SVD, LDA

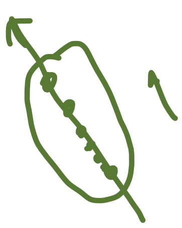
PCA (Principal Component Analysis)

$\{x_n\}_{n=1 \ldots N}$ dim = D M < D



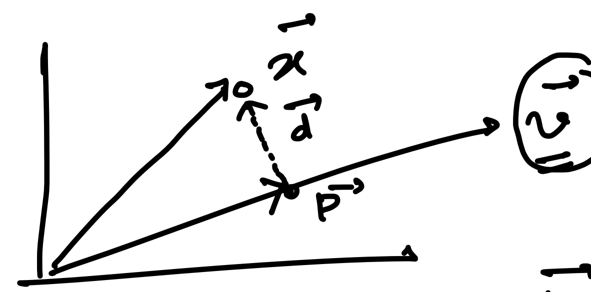


Choose u_1 to project $\{x_n\}$ s.t. the variance of $\{x_n\}$ on u_1 is maximized.



Vector projection

$$\vec{p} = \frac{\vec{x} \cdot \vec{v}}{|v|} \cdot \hat{v}$$



unit vector: $\hat{u} = \frac{\vec{v}}{\|v\|}$

$$\vec{p} + \vec{d} = \vec{x}$$
$$\vec{d} = \vec{x} - \vec{p}$$
$$= \vec{x} - K\hat{u}$$

$$\bar{p} \cdot \bar{d} = 0$$
$$(K\hat{u}) \cdot (\vec{x} - K\hat{u}) = 0$$
$$\Rightarrow \cancel{K}\, \vec{x} \cdot \hat{u} - K^{\cancel{2}}\, \hat{u} \cdot \hat{u} = 0$$
$$\Rightarrow \vec{x} \cdot \hat{u} = K$$

$$P = K\hat{u}$$
$$= (\vec{x} \cdot \hat{u})\, \hat{u}$$
$$= \left(\frac{\vec{x} \cdot \vec{v}}{\|v\|} \right) \hat{u}$$

mean: $\{x_n\} = \bar{x}$

mean $\{x_n\}$ on u_1
$= u_1^T \bar{x} \cdot \hat{u}$

$$\text{Proj}(x_1) = u_1^T x_1$$

Variance: $\frac{1}{N} \sum_{n=1}^{N} \left(u_1^T x_n - u_1^T \bar{x} \right)^2$

$$= \frac{1}{N} \sum \left[u_1^T (x_n - \bar{x}) \right]^2$$

$$= \frac{1}{N} \sum_{n=1}^{N} \left[u_1^T (x_n - \bar{x}) (x_n - \bar{x})^T u_1 \right]$$

$$= u_1^T \left[\frac{1}{N} \sum_{n=1}^{N} (x_n - \bar{x}) (x_n - \bar{x})^T \right] u_1$$

$$\max = \boxed{u_1^T S u_1} =$$ S = covariance matrix

$\max u_1^T S u_1$ s.t. u_1 is a unit vector.

$u_1 u_1^T = 1$

$$\mathcal{L} = u_1^T S u_1 - \lambda (u_1 u_1^T - 1)$$ λ: Lagrange multiplier.

$$= u_1^T S u_1 + \lambda (1 - u_1 u_1^T)$$

$$\frac{\partial \mathcal{L}}{\partial u_1} = 2 u_1 S - 2 \lambda u_1 = 0$$

$$\underline{S u_1 = \lambda u_1}$$

inner product with u_1^T

$$\underline{u_1^T S u_1} = \lambda \underline{u_1^T u_1}$$

$$= \lambda$$

Rank of $A \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \dots & a_n \\ | & | & | & & | \end{bmatrix}_{m \times n} = 2$

$a_3 = c_3^1 a_1 + c_3^2 a_2$

$a_4 = c_4^1 a_1 + c_4^2 a_2$

$\vdots$

$M = \begin{bmatrix} c_3^1 & c_3^2 \\ c_4^1 & c_4^2 \\ \vdots & \vdots \end{bmatrix}$

$$A = \boxed{\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}_{m \times 2} \begin{bmatrix} M \end{bmatrix}^T_{2 \times n}}$$

$\underline{2m + 2n < m \times n}$

- Singular Value Decomposition -

Eigen Decomposition.

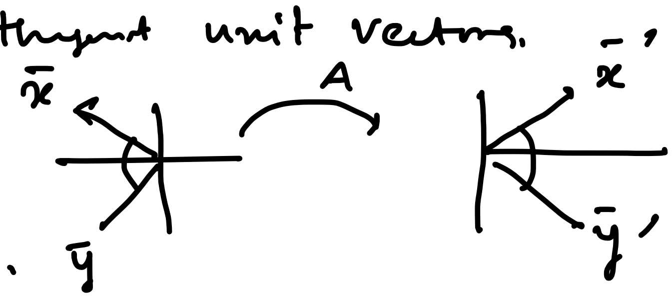
$A|_{m \times m} = U \Sigma U^T$

U is orthonormal.

$U^T = U^{-1}$

$$= U \Sigma U^{-1}$$

Let us assume that $\{\bar{x}, \bar{y}\}$ are orthogonal unit vectors.



$$\begin{cases} A \cdot \bar{x} = \bar{x}' = a_1 \bar{v}_1 \\ A \cdot \bar{y} = \bar{y}' = a_2 \bar{v}_2 \end{cases}$$

$\bar{v}_1, \bar{v}_2$ are unit vectors.

$$A [\bar{x}, \bar{y}] = [a_1 \bar{v}_1, a_2 \bar{v}_2]$$
$$= [\bar{v}_1, \bar{v}_2] \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

$U = [\bar{x}, \bar{y}]$

$V = [\bar{v}_1, \bar{v}_2]$

Σ

$$\Rightarrow AU = V\Sigma$$
$$\Rightarrow A = V \Sigma U^T$$

$$A^T A = (V \Sigma U^T)^T (V \Sigma U^T)$$
$$= U \Sigma^T V^T V \Sigma U^T$$
$$= U \Sigma^2 U^T$$

$$A A^T = V \Sigma^2 V^T$$

$$\boxed{A = V \Sigma U^T}$$

$$\Sigma = \begin{bmatrix} \sqrt{\sigma} & & & \\ & \sqrt{\sigma} & & \\ & & \sqrt{\sigma} & \\ & & & \sqrt{\sigma} \end{bmatrix}$$

SVD is trying to find a direction of maximum stretching

$$\arg\max_x |Ax| \longrightarrow |x| = 1$$
$$= |Ax|^2$$
$$= (Ax)^T (Ax)$$
$$= x^T A^T A x$$

$A^T A$

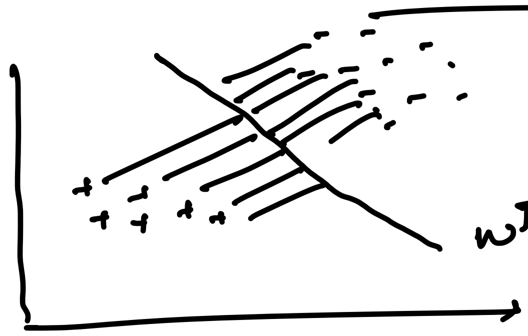
$(A^T A) x = \lambda x$

$x^T (A^T A) x = \lambda x^T x$

$= \lambda$

λ = principal eigen vector of $A^T A$

Linear Discriminant Analysis (LDA)



Between-class separation maximized

in-class separation minimized

$C_0 \qquad C_1$

$n_0 \qquad n_1$

$n = n_0 + n_1$

$$\text{mean of } C_0 = \frac{1}{n_0} \sum_{x_i \in C_0} x_i = M_0$$

$$\text{mean of } C_1 = \frac{1}{n_1} \sum_{x_i \in C_1} x_i = M_1$$

$w^T M_0 = m_0$ = mean for C_0

$w^T M_1 = m_1$ = mean for C_1

$$\sigma_0^2 = \sum_{x_i \in C_0} (w^T x_i - m_0)^2$$