Chapter 2: The Language of Bits

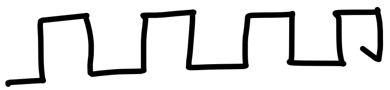
Basic Computer Architecture

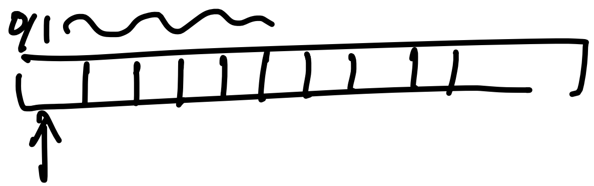
Outline

* Boolean Algebra
* Positive Integers
* Negative Integers
* Floating-Point Numbers
* Strings

What does a Computer Understand ?

* Computers do not understand natural human languages, nor programming languages
* They only understand the language of bits





Bit	0 or 1
Byte	8 bits
Word	4 bytes
kiloByte	1024 bytes
megaByte	10^6 bytes

$1\ KG = 10^3\ gm$

$1\ KB = 2^{10}$ bite $\approx 10^3$ bite

1024 → 1000

$\Rightarrow 2^{10}$ 2^{20}

$\Rightarrow 10^3 \times 10^3 = 10^6$

$2^{10} \times 2^{10}$

Review of Logical Operations

* A + B (A or B)

OR

A OR B

A || B

A | B

A	B	A + B
0	0	0
1	0	1
0	1	1
1	1	1

0/1

= Truth Table

* A.B (A and B)

AND

A B

NOT

A	B	A.B
0	0	0
1	0	0
0	1	0
1	1	1

Review of Logical Operations - II

A	B	A NAND B
0	0	1
1	0	1
0	1	1
1	1	0

A	B	A NOR B
0	0	1
1	0	0
0	1	0
1	1	0

$\bar{A}$

~A

A!

* NAND and NOR operations
* These are universal operations. They can be used to implement any Boolean function.

Review of Logical Operations

- XOR Operation : $(A \oplus B)$

A	B	A XOR B
0	0	0
1	0	1
0	1	1
1	1	0

How many truth tables we can build?

Review of Logical Operations

- NOT operator
  - Definition: $\overline{0} = 1$, and $\overline{1} = 0$
  - Double negation: $\overline{\overline{A}} = A$, NOT of (NOT of A) is equal to A itself
- OR and AND operators
  - Identity: $A + 0 = A$, and $A.1 = A$
  - Annulment: $A + 1 = 1$, $A.0 = 0$

* Idempotence: A + A = A, A.A = A, The result of computing the OR and AND of A with itself is A.
* Complementarity: $A + \overline{A} = 1$, $A.\overline{A} = 0$
* Commutativity: A + B = B + A, A.B = B.A, the order of Boolean variables does not matter
* Associativity: A+(B+C) = (A+B)+C, A.(B.C) = (A.B).C, similar to addition and multiplication.
* Distributivity: A.(B + C) = A.B + A.C, A+ (B.C) = (A+B). (A+C) → Use this law to open up parantheses and simplify expressions

$A + B$

$A + (B + C)$
$\shortparallel$
$(A+B) + C$

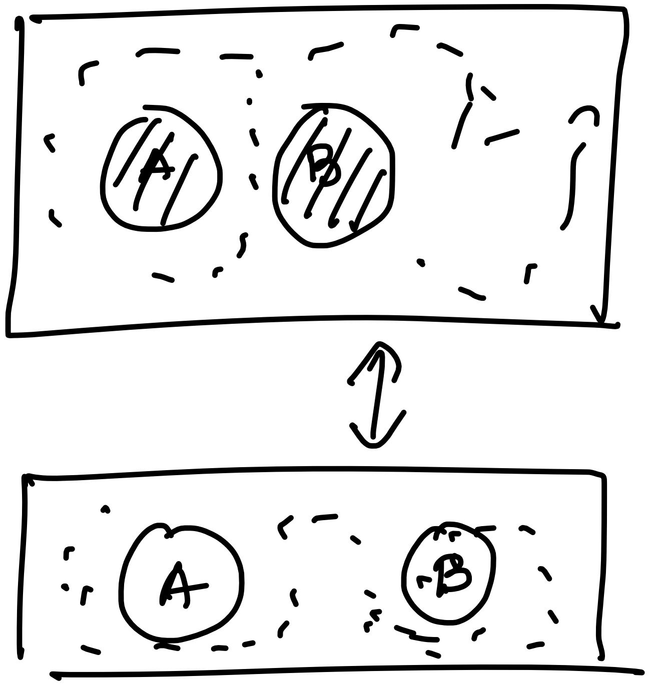
$x(y+z) = xy + xz$

De Morgan's Laws

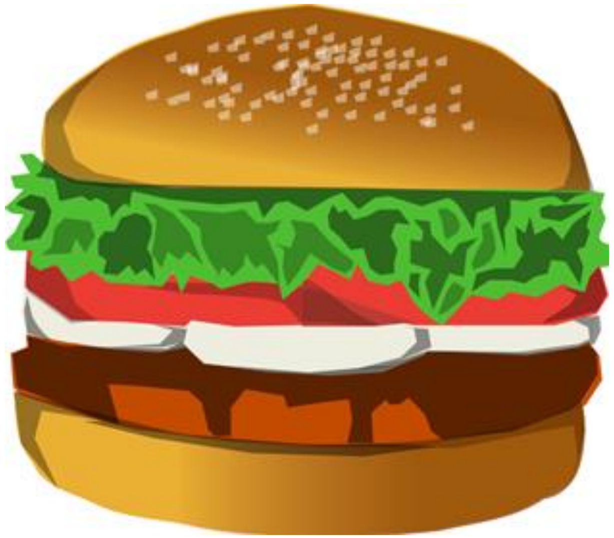
* Two very useful rules

$$\overline{A + B} = \overline{A}.\overline{B}$$

$$\overline{A.B} = \overline{A} + \overline{B}$$



Consensus Theorem



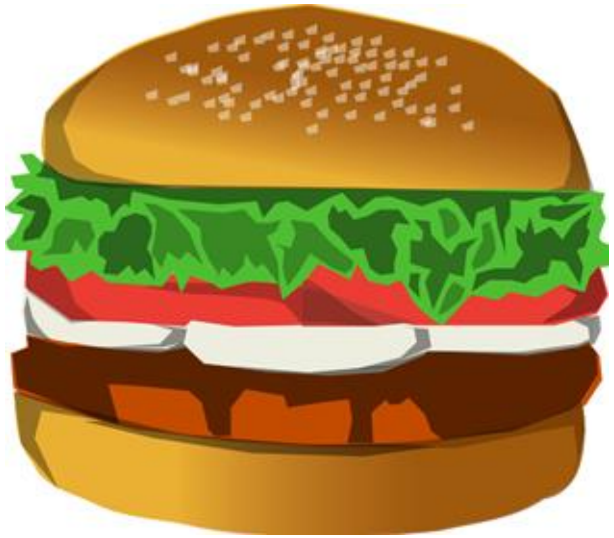
* Prove :
  * $X.Y + \bar{X}.Z + Y.Z = X.Y + \bar{X}.Z$

$X.Y + \bar{X}Z + YZ(X + \bar{X})$

$XY + \bar{X}Z + XYZ + \bar{X}YZ$

$XY(1+Z) + \bar{X}Z(1+Y) = XY + \bar{X}Z$

Consensus Theorem



- Prove :
  - $X.Y + \overline{X}.Z + Y.Z = X.Y + \overline{X}.Z$

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- Boolean Algebra
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- Negative Integers
- Floating Point Numbers
- Strings

Representing Positive Integers

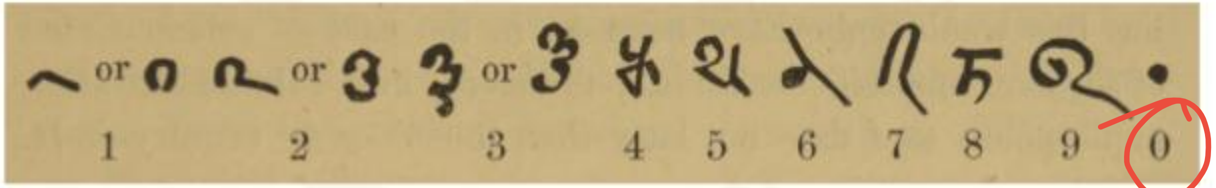
* Ancient Roman System

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

place-value representation

* Issues :
  * There was no notion of 0
  * Very difficult to represent large numbers
  * Addition, and subtraction (very difficult)

Indian System (place –value system)



Bakshali numerals, 7th century AD

5301 = place value.

* Uses the place value system

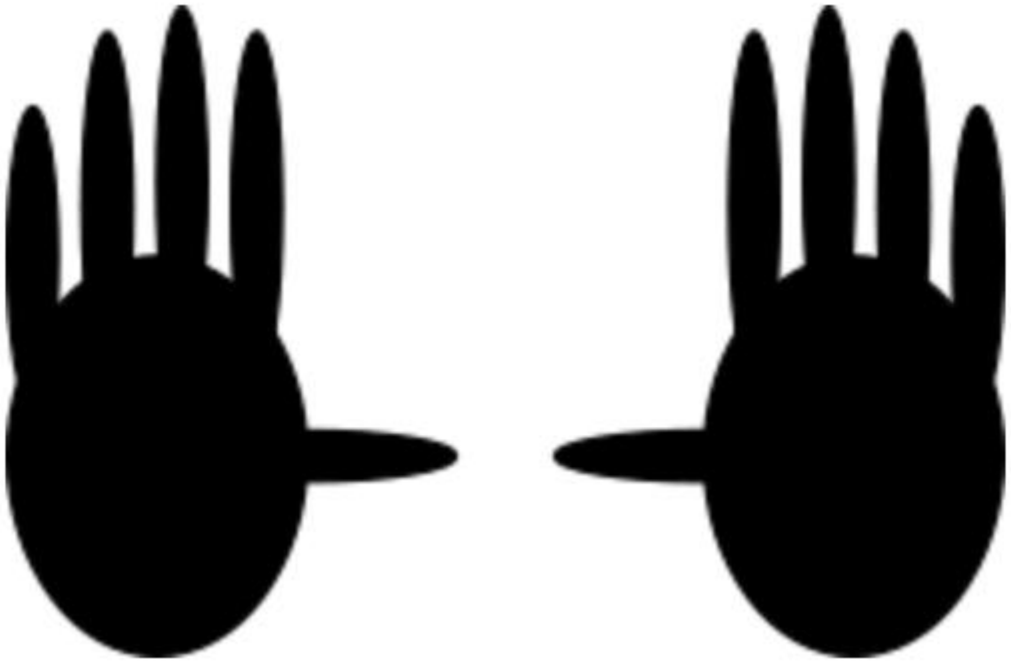
$$5301 = 5 * 10^3 + 3 * 10^2 + 0 * 10^1 + 1*10^0$$

base value

Example in base 10

Number Systems in Other Bases

* Why do we use base 10 ?
  * because ...



What if we had a world in which …

* People had only two fingers.

Binary Number System

* They would use a number system with base 2.

Number in decimal	Number in binary
5	101
100	1100100
500	111110100
1024	10000000000

19

$= 10011$

MSB

LSB

Least significant bit

$2 | 5$
$2 | 2 - 1$
$1 - 0$

$5 = \underline{101}$

$= 10^0 \times 5$

$\cancel{1 \times 2}$

$2^2 \times 1 + 2^1 \times 0 + 2^0 \times 1$

base value = 2

place value starts from 0

MSB and LSB

* MSB (Most Significant Bit) → The leftmost bit of a binary number. E.g., MSB of 1110 is 1
* LSB (Least Significant Bit) → The rightmost bit of a binary number. E.g., LSB of 1110 is 0

Hexadecimal and Octal Numbers

* Hexadecimal numbers
  * Base 16 numbers – 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F
  * Start with 0x
* Octal Numbers = base=8
  * Base 8 numbers – 0,1,2,3,4,5,6,7
  * Start with 0

Base = 16

0
1
2
3
4
5
6
7
8
9
A — 10
B — 11
C — 12
D — 13
E — 14
F — 15

1 Why Octal given that we have Hexa ?

Examples

Convert 110010111 to the octal format : $\underbrace{110}_{6}\ \underbrace{010}_{2}\ \underbrace{111}_{7}$ = 0627

Convert 111000101111 to the hex format : $\underbrace{1110}_{14 = E}\ \underbrace{0010}_{2}\ \underbrace{1111}_{15 = F}$ = 0xE2F

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7$

$$N = \underbrace{(a_0 \times 2^0 + a_1 \times 2^1 + a_2 \times 2^2)}_{b_0} + (a_3 2^3 + a_4 2^4 + a_5 2^5) + \cdots$$

$$= b_0 + 2^3 (a_3 \times 2^0 + a_4 \times 2^1 + a_5 2^2) + 2^6 (a_6 \times 2^0 + a_7 2^1 + a_8 2^2)$$

$$= 2^0 \underbrace{(0-7)} + \underbrace{2^3}_{8} (0-7) + (2^3)^2 + (0-7) + \cdots + \cdots$$

$$= 8^0 b_0 + 8^1 b_1 + 8^2 b_2 8 + \cdots$$

Examples

$\boxed{(X)_2 \rightarrow (Y)_5}$

Convert 110010111 to the octal format : $\underbrace{110}\,\underbrace{010}\,\underbrace{111}$ = 0627

Convert 111000101111 to the hex format : $\underbrace{1110}\,\underbrace{0010}\,\underbrace{1111}$ = 0xE2F

2. Can we convert a 2^n- base system to a base system which can't be represented by 2^m; e.g.; binary $\rightarrow (Y)_{\cancel{3}\,5}$ without converting to decimal

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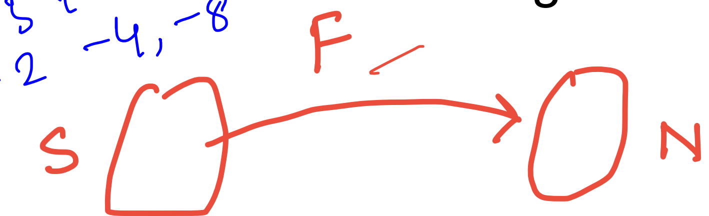
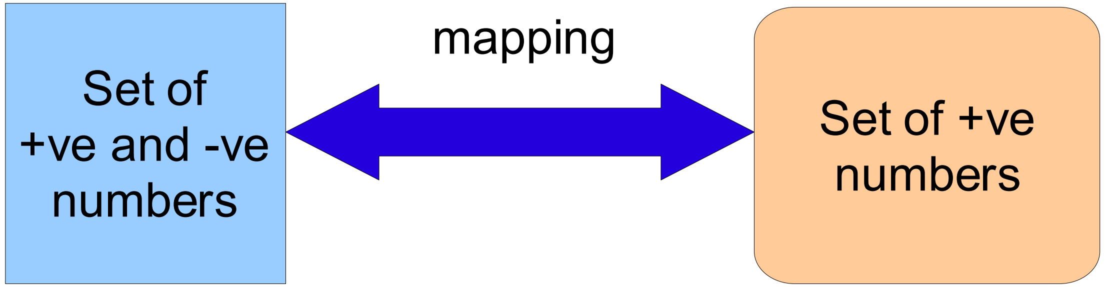
Representing Negative Integers

- Problem
  - Assign a binary representation to a negative integer
  - Consider a negative integer, S
  - Let its binary representation be : $x_n x_{n-1} \ldots x_2 x_1$ (x_i=0/1)
  - We can also expand it to represent an unsigned, +ve, number, N
  - If we interpret the binary sequence as :
    - An unsigned number, we get N
    - A signed number, we get S

+5 +3
-2 -4, -8



* We need a mapping :
  * F : S → N (mapping function)
  * S → set of numbers (both positive and negative – signed)
  * N → set of positive numbers (unsigned)

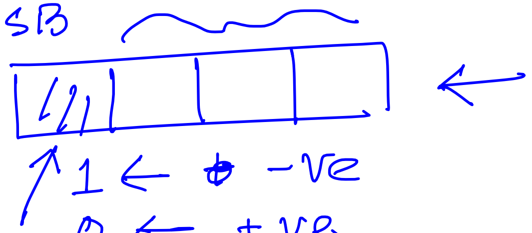


Properties of the Mapping Function

- Preferably, needs to be a one to one mapping
- All the entries in the set, S, need to be mapped
- It should be easy to perform addition and subtraction operations on the representation of signed numbers
- Assume an n bit number system

$|-5| = 5$

MSB



$1 \leftarrow$ ~~0~~ $-ve$

$0 \leftarrow +ve$

$$\text{SgnBit}(u) = \begin{cases} 1\ , & u < 0 \\ 0\ , & u >= 0 \end{cases}$$

Sign-Magnitude Base Representation

0/1

$$\boxed{F(u) = SgnBit(u) * 2^{n-1} + |u|}$$

sign bit	|u|

-5 $n = 4$

$n - 1 = 8$

1	1	0	1

1×2^3 $\underbrace{|5|}$ → 13

+4

0	1	0	0

$= 4$

- Examples :
  - -5 in a 4 bit number system : 1101
  - 5 in a 4 bit number system : 0101
  - -3 in a 4 bit number system : 1011

Problems

* There are two representations for 0
  * 000000
  * 100000
* Addition and subtraction are difficult
* The most important takeaway point :
  * Notion of the sign bit

1's Complement Representation

$S \rightarrow N$

$\uparrow$ F

$$F(u) = \begin{cases} u, u \geq 0 \\ \sim(|u|) or (2^n - 1 - |u|), u < 0 \end{cases}$$

$n = 4$

$2^4 = 16$

$2^4 - 1 = 15$

* Examples in a 4 bit number system
  * 3 → 0011
  * -3 → 1100 = 12
  * 5 → 0101
  * -5 → 1010

$+3 = 0011$

$-3 = \sim(+3)$

$= 1100$

Notion of sign bit also exists

$F(-3) = 2^4 - 1 - |-3|$

$= 15 - 3$

$= 12$

$= 1100$

Problems

* Two representations for 0
  * 0000000
  * 1111111
* Easy to add +ve numbers
* Hard to add -ve numbers
* Point to note :
  * The idea of a complement

$3 = 0011$
$+\,2 = 0010$
0101
$= 5$

$3 + (-2) = 1$

0011
$+\,1101$
10000
$+1$
0001

Bias Based Approach

$$\boxed{F(u) = u + \underbrace{bias}_{=7}}$$

- Consider a 4 bit number system with bias equal to 7
  - -3 → 0100
  - 3 → 1010
- $\boxed{F(u+v) = F(u) + F(v) - bias}$
- Add and Sub are also easy
- Multiplication is difficult

$-3 = -3+7 = +4$

$+2 = +2+7 = +9$

$+3 = +3+7 = 10$ 1010

$-3 = -3+7 = 4$ -0100

1110

14

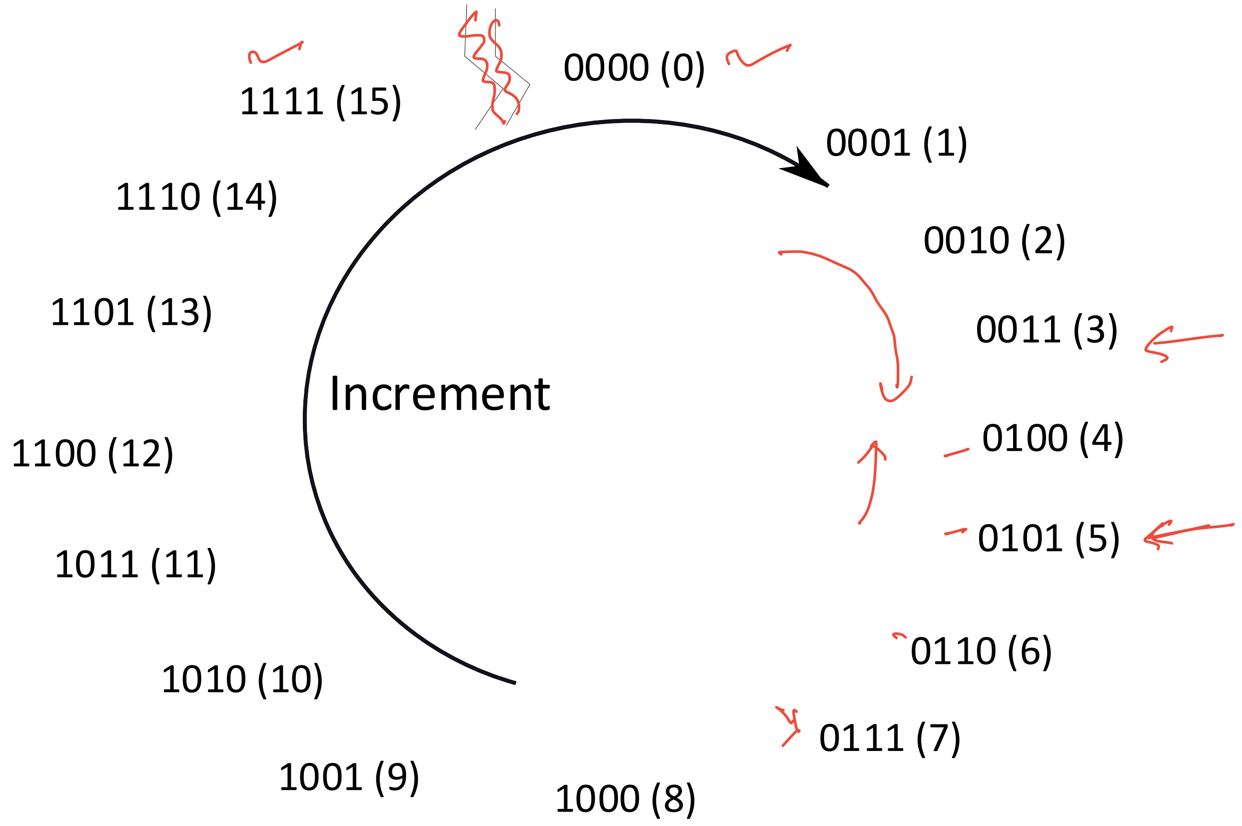
-7

7

$F(u-v) = F(u) - F(v) + bias.$

$F(3-3) = F(3) - F(3) + 7 = 7 = 0$

The Number Circle

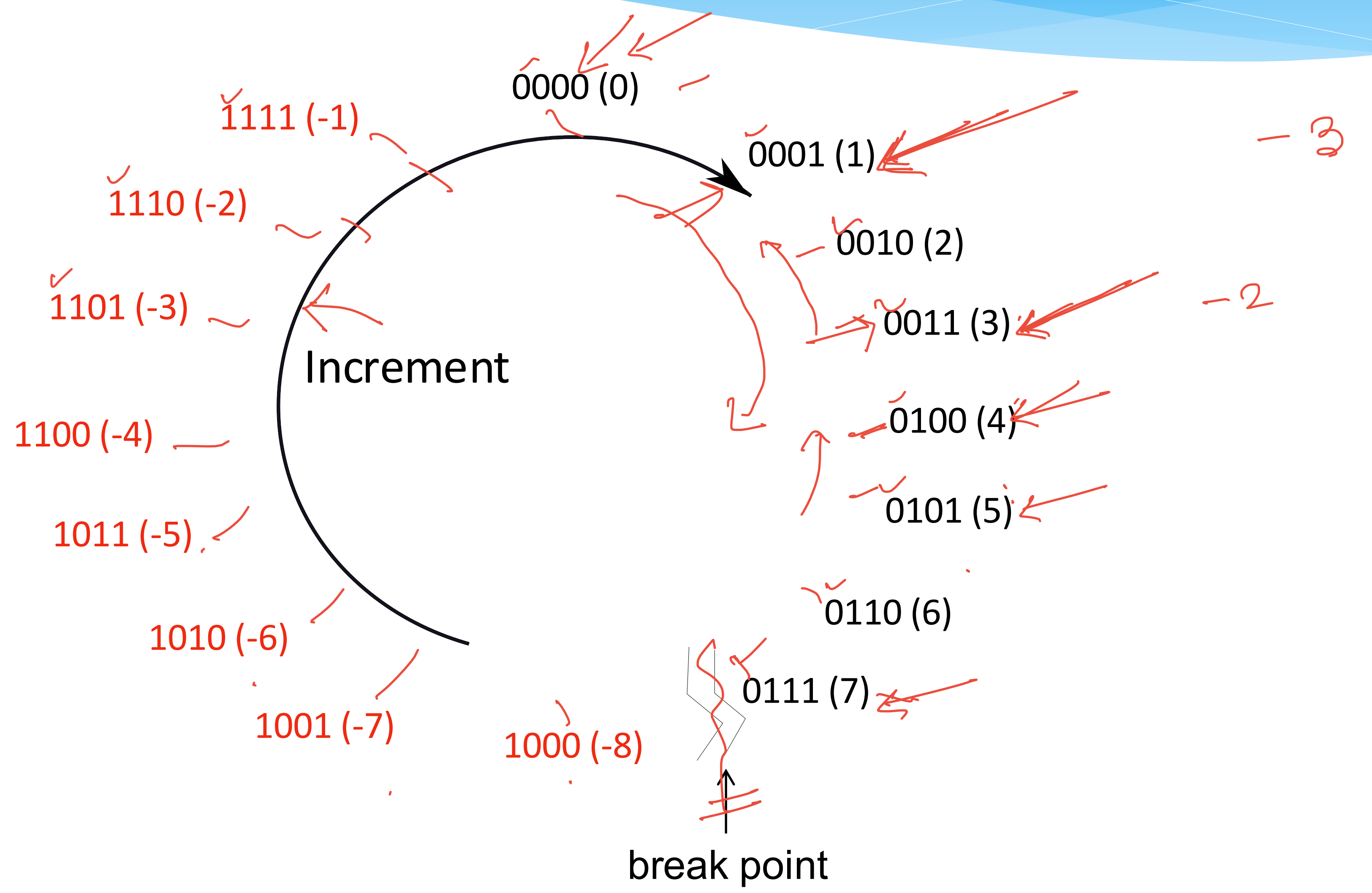


4 bits

Overflow

Clockwise: increment
Anti-clockwise: decrement

Number Circle with Negative Numbers



Using the Number Circle

- To add M to a number, N
  - locate N on the number circle
  - If M is +ve
    - Move M steps clockwise
  - If M is -ve
    - Move M steps anti-clockwise, or 2^n – M steps clockwise
  - If we cross the break-point
    - We have an overflow / underflow
    - The number is too large/ too small to be represented

2

n = 16

2's Complement Notation

$$F(u) = \begin{cases} u, & 0 \le u \le 2^{n-1} - 1 \\ 2^n - |u|, & -2^{n-1} \le u < 0 \end{cases}$$

$n = 4$

$2^3 - 1$

$-2^{n-1} \quad - 0$

$-8 \to 0$

$16 - 2$

$-2 = 14$

$-3 = 13$

- F(u) is the index of a point on the number circle. It varies from 0 to 2^n - 1
- Examples
  - 4 → 0100
  - -4 → 1100
  - 5 → 0101
  - -3 → 1101

Properties of the 2's Complement Notation

* Range of the number system :
  * $-2^{(n-1)}$ to $2^{n-1} - 1$

    -8 $+7$
* There is a unique representation for 0 → 000000
* msb of F(u) is equal to SgnBit(u)
  * Refer to the number circle
  * For a +ve number, $F(u) < 2^{(n-1)}$. MSB = 0
  * For a -ve number, $F(u) >= 2^{(n-1)}$. MSB = 1

Properties - II

- Every number in the range $[-2^{(n-1)}, 2^{(n-1)} - 1]$
  - Has a unique mapping
  - Unique point in the number circle

$a - b \bmod 2^n = 0 \quad n = 4$

- $a \equiv b \rightarrow (a = b \bmod 2^n)$

$+2 \equiv +18$

- $\equiv$ means same point on the number circle

- $F(-u) \equiv 2^n - F(u)$
  - Moving F(u) steps counter clock wise is the same as moving $2^n - F(u)$ steps clockwise from 0

Prove : $F(u+v) \equiv F(u) + F(v)$

- Start at point u
  - Its index is F(u)
  - If v is +ve,
    - move v points clockwise. We arrive at F(u+v).
    - Its index is equal to (F(u) + v) mod 2^n.
    - Since v = F(v), we have F(u+v) = (F(u) + F(v)) mod 2^n

$u = 2$
$v = 3$
$F(2+3) = F(2) + F(3)$
$F(5) = F(2) + F(3)$
$= 0010 + 0011$
$= 0101$

Prove : $F(u+v) \equiv F(u) + F(v)$

- If v is -ve,
  - move |v| points anti-clockwise.
  - Same as moving $2^n - |v|$ points clockwise.
  - We arrive at F(u+v).
  - $F(v) = 2^n - |v|$
  - The index – F(u+v) – is equal to:
    - $(F(u) + 2^n - |v|) \bmod 2^n = (F(u) + F(v)) \bmod 2^n$

$u = 2$
$v = -4$
$F(2+(-4)) = F(2) + F(-4)$
$= 0010$
$+ 1100$
$= 1110$
$= F(-2)$

Subtraction

- $F(u-v) \equiv F(u) + F(-v)$

  $\equiv F(u) + 2^n - F(v)$

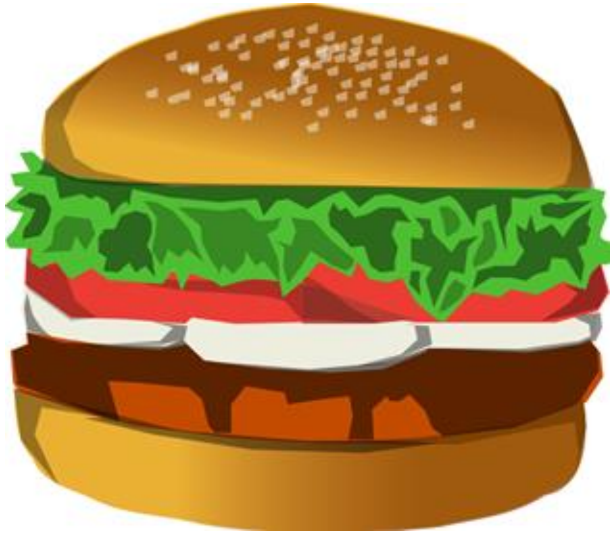
$F(3-5) = F(3) + F(-5)$

$= 3 + 16 - 5$

$= 14 = F(-2)$

- Subtraction is the same as addition
- Compute the 2's complement of F(v)

Prove that :



* Prove that :

$$F(u*v) \equiv \mathbf{F(u) * F(v)}$$

$u = 2$, $v = -3$

$F(2 \times (-3))$

$\equiv F(2) \times F(-3)$

$= 2 \times 13$

$= 26 \bmod 16$

$= 10 = F(-6)$

1. if u and v and +ve, it is trivial.

2. if u and v are −ve, $u = -|u|$ $v = -|v|$

$$F(u) \times F(v) = \left(2^n - F(|u|)\right) \times \left(2^n - F(|v|)\right)$$

$$= \cancel{2^{2n}} - \cancel{2^n\left(F(|u|) + F(|v|)\right)} + F(|u|) \times F(|v|)$$

$$= \cancel{0} \quad F(|u|) \times F(|v|)$$

$$= F(|u| \times |v|) = F(u \times v)$$

3. u is +ve
v is −ve

$u = |u|$
$v = -|v|$

$$F(u) \times F(v) = F(u) \times \left(2^n - F(|v|)\right)$$

$$= 2^n \cancel{F(u)} - F(u) \times F(|v|)$$

$$= -F(u) \times F(|v|)$$

$$= -F(u \times |v|) \qquad u \geq 0 \quad |v| \geq 0$$

$$= 2^n - F(u \times |v|)$$

$$= F(-(u \times |v|))$$

$$= F(u \times (-|v|)) = F(u \times v)$$

$$F(-u) = 2^n - F(|u|)$$

negative rule

Computing the 2's Complement

* $2^n - u$

$2^n - 1$

  $= \boxed{2^n - 1 - u} + 1$

  $= \sim u + 1$

  * $\sim u$ (1's complement)

* 1's complement of 0100

$$\begin{array}{r} 1111 \\ -\ 0100 \\ \hline 1011 \end{array}$$

2's complement of 0100

$$\begin{array}{r} 1011 \\ +\ 0001 \\ \hline 1100 \end{array}$$

Sign Extension

4 8

- Convert a n bit number to a m bit 2's complement number (m > n)

$+4 = 0100$

$0000\ 0100$

- +ve
  - Add (m-n) 0s in the msb positions
  - Example, convert 0100 to 8 bits → 0000 0100

1100

$\boxed{1111\ 1100} = -252$

- -ve
  - F(u) = 2^n – |u| (n bit number) system
  - Need to calculate F'(u) = 2^m -|u|

Sign Extension - II

$m > n$

$2^m - 2^n$

$= 2^n + 2^{n+1} + \dots + 2^{m-1}$

$2^m - 2^n$

$= 2^n(2^{m-n} - 1)$

$2^k - 1 = 1 + 2 + 2^2 + \dots + 2^{k-1}$

$= 2^n(1 + 2 + 2^2 + \dots + 2^{m-n-1})$

$= 2^n(1 + 2 + 2^2 + \dots + 2^{m-n-1})$

$= 2^n + 2^{n+1} + \dots + 2^{m-1}$

* $2^m - u - (2^n - u)$

$= \boxed{2^m - 2^n}$

$= 2^n + 2^{(n+1)} + \dots + 2^{(m-1)}$

$= \underbrace{1111}_{m-n}\underbrace{0000}_{n}$ $+ (-2)$

$m = 4 \quad n = 2$

$2^2 + 2^3$

$2^4 - 2^2$

$= 16 - 4$

$= 12$

$= 2^2 + 2^3$

$4 - 8$

$$\begin{array}{r} 1111\,0000 \\ +\ 1100 \\ \hline 1111\,1100 \end{array}$$

$F'(u) = F(u) + 2^m - 2^n$

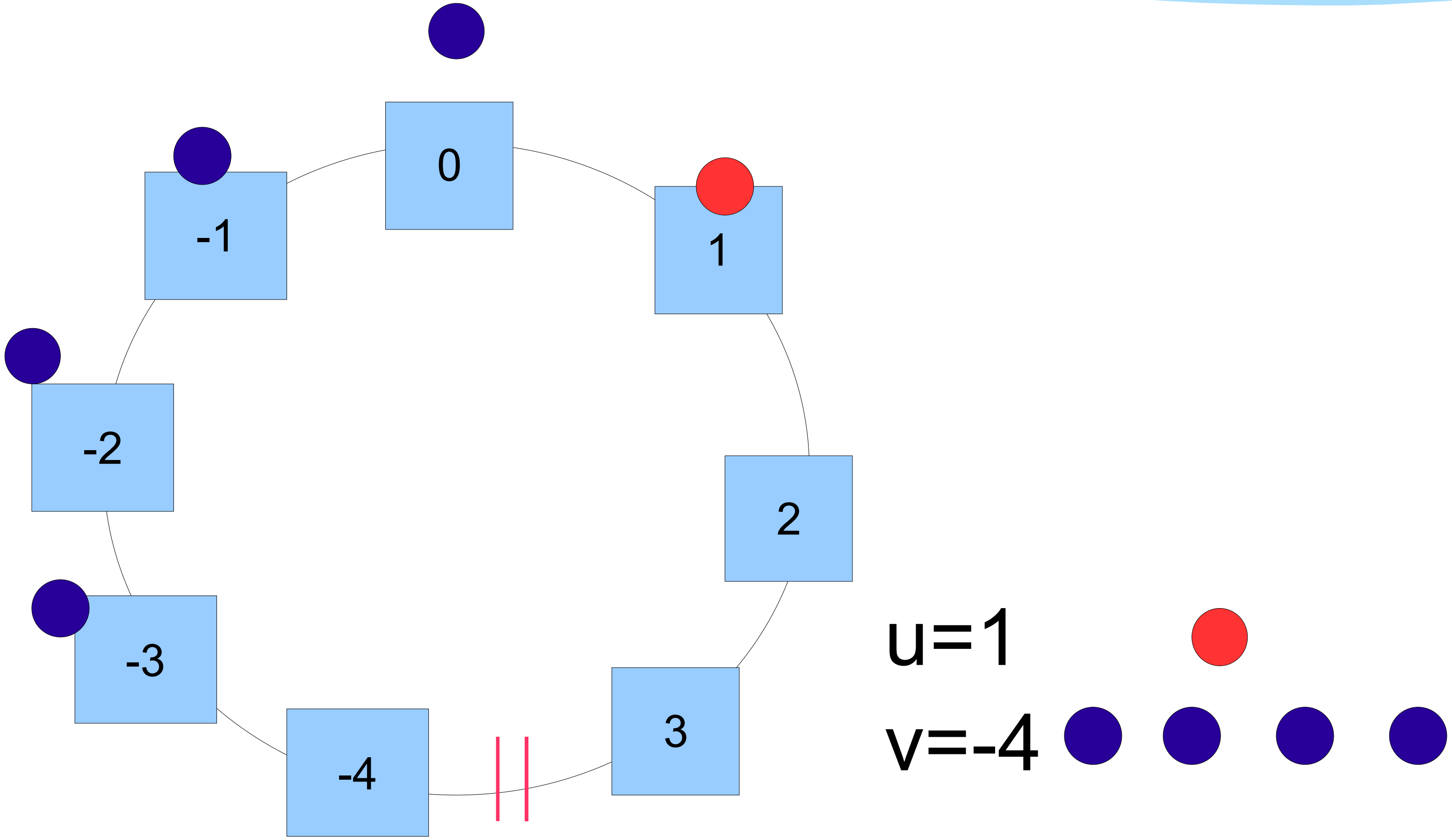
Sign Extension - III

* To convert a negative number :
  * Add (m-n) 1s in the msb positions
* In both cases, extend the sign bit by :
  * (m-n) positions

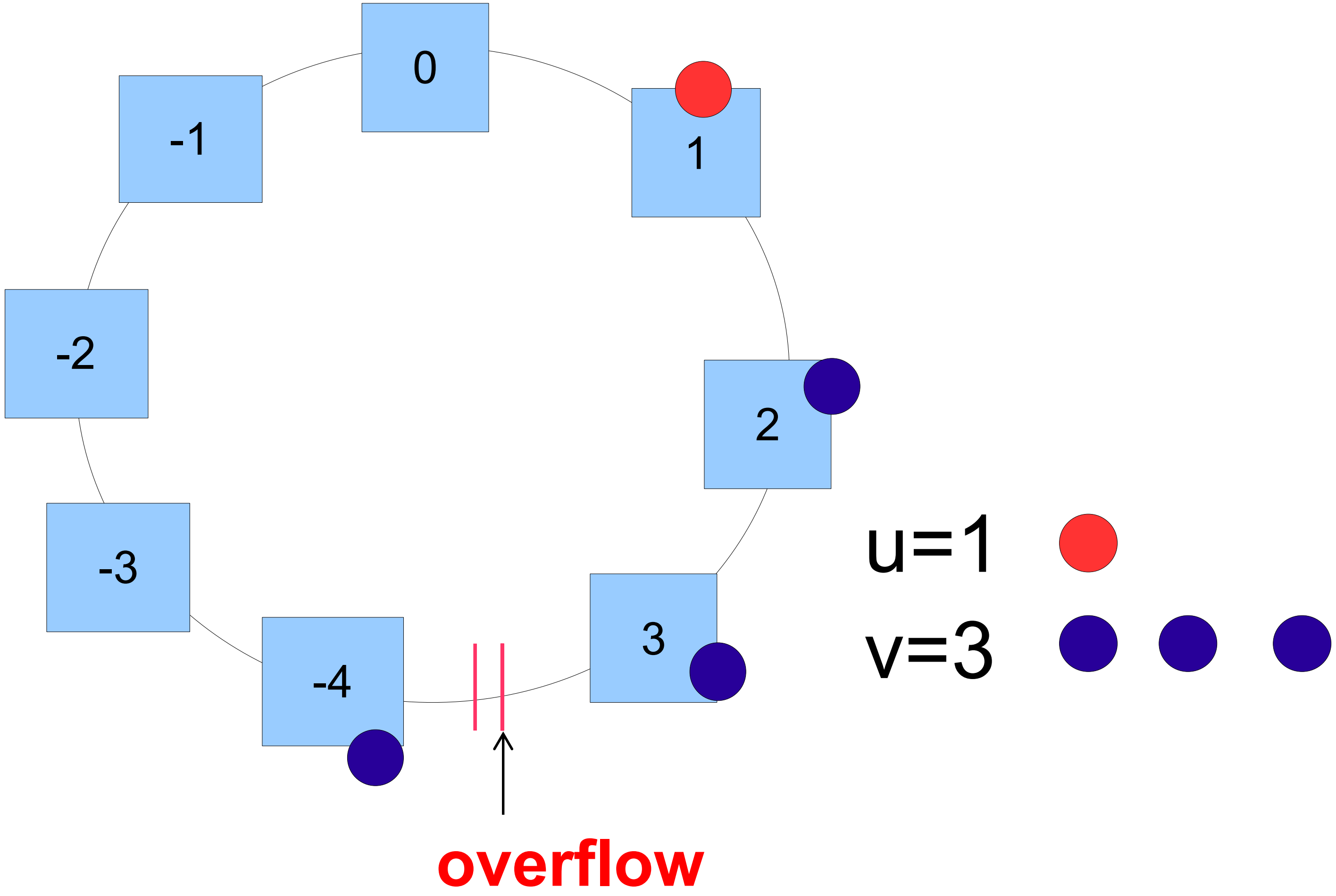
The Overflow Theorem

- Add : u + v
- If $uv < 0$, there will never be an overflow
- Let us go back to the number circle
  - There is an overflow only when we cross the break-point
- If $uv = 0$, one of the numbers is 0 (no overflow)
- If $uv > 0$, an overflow is possible

Number Circle: uv < 0



Number Circle: uv > 0



Conditions for an Overflow

- uv <= 0
  - Never
- uv > 0 (u and v have the same sign)
  - The sign of the result is different from the sign of u

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Floating-Point Numbers

- What is a floating-point number ?
  - 2.356
  - 1.3e-10
  - -2.3e+5
- What is a fixed-point number ?
  - Number of digits after the decimal point is fixed
  - 3.29, -1.83

Generic Form for Positive Numbers

- Generic form of a number in base 10

$$A = \sum_{i=-n}^{n} x_i 10^i$$

- Example :
  - $3.29 = 3 * 10^0 + 2*10^{-1} + 9*10^{-2}$

Generic Form in Base 2

* Generic form of a number in base 2

$$A = \sum_{i=-n}^{n} x_i 2^i$$

Number	Expansion
0.375	$2^{-2} + 2^{-3}$
1	2^0
1.5	$2^0 + 2^{-1}$
2.75	$2^1 + 2^{-1} + 2^{-2}$
17.625	$2^4 + 2^0 + 2^{-1} + 2^{-3}$

Binary Representation

* Take the base 2 representation of a floating-point (FP) number
* Each coefficient is a binary digit

Number	Expansion	BinaryRepresentation
0.375	$2^{-2} + 2^{-3}$	0.011
1	2^{0}	1.0
1.5	$2^{0} + 2^{-1}$	1.1
2.75	$2^{1} + 2^{-1} + 2^{-2}$	10.11
17.625	$2^{4} + 2^{0} + 2^{-1} + 2^{-3}$	10001.101

Normalized Form

- Let us create a standard form of all floating point numbers

$$A = (-1)^S * P * 2^X, (P = 1 + M, 0 \leq M < 1, X \in Z)$$

- S → sign bit, P → significand
- M → mantissa, X → exponent, **Z** → set of integers

Examples (in decimal)

* 1.3827 * 1e-23
  * Significand (P) = 1.3827
  * Mantissa (M) = 0.3827
  * Exponent (X) = -23
  * Sign (S) = 0
* -1.2*1e+5
  * P = 1.2 , M = 0.2
  * S = 1, X = 5

IEEE 754 Format

- General Principles
  - The significand is of the form : 1.xxxxx
  - No need to waste 1 bit representing (1.) in the significand
  - We can just save the mantissa bits
  - Need to also store the sign bit (S), exponent (X)

IEEE 754 Format - II

Sign(S)	Exponent(X)	Mantissa(M)
1	8	23

* sign bit – 0 (+ve), 1 (-ve)
* exponent, 8 bits
* mantissa, 23 bits

Representation of the Exponent

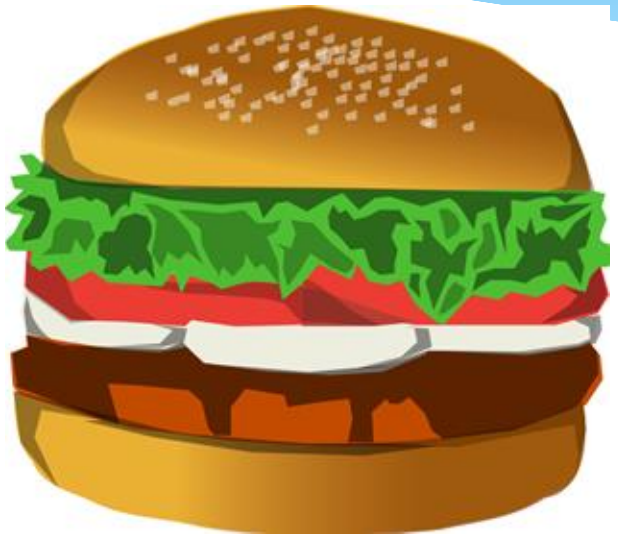
- Biased representation
  - bias = 127
  - E = X + bias
- Range of the exponent
  - 0 – 255 ⟷ -127 to +128
- Examples :
  - X = 0, E = 127
  - X = -23, E = 104
  - X = 30 , E = 157

Normal FP Numbers

- Have an exponent between -126 and +127
- Let us leave the exponents : -127, and +128 for special purposes.

$$A = (-1)^S * P * 2^{E-bias}$$

$$(P = 1 + M, 0 \le M < 1, X \in Z, 1 \le E \le 254)$$



* What is the largest +ve normal FP number ?
* What is the smallest –ve normal FP number ?

Special Floating Point Numbers

E	M	Value
255	0	∞ if $S = 0$
255	0	$-\infty$ if $S = 1$
255	$\neq 0$	NAN(Not a number)
0	0	0
0	$\neq 0$	Denormal number

- NAN + x= NAN   1/0 = ∞
- 0/0 = NAN   -1/0 = -∞
- $\sin^{-1}(5)$ = NAN

Denormal Numbers

```
f = 2^(-126);
g = f/2;
if (g == 0)
  print ("error");
```

* Should this code print "error" ?
* How to stop this behaviour ?

Denormal Numbers - II

$$A = (-1)^S * P * 2^{-126}$$

$$(P = 0 + M, 0 \le M < 1)$$

- Significand is of the form : 0.xxxx
- E = 0, X = -126 (why not -127?)
- Smallest +ve normal number : 2^{-126}
- Largest denormal number :
  - 0.11…11 * 2^{-126} = $(1 – 2^{-23})*2^{-126}$
    - $=2^{-126} - 2^{-149}$

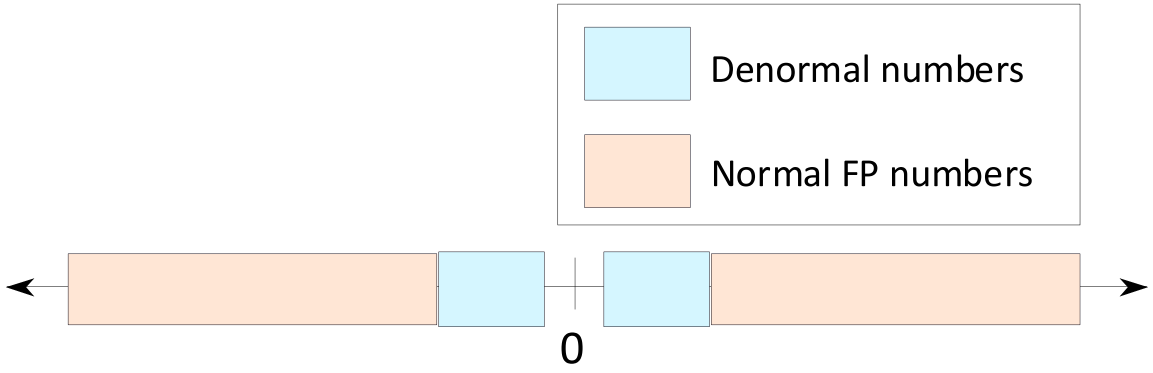
Example

Find the ranges of denormal numbers.

Answer

- For positive denormal numbers, the range is $[2^{-149}, 2^{-126} - 2^{-149}]$
- For negative denormal numbers, the range is $[-2^{-149}, -2^{-126} + 2^{-149}]$

Denormal Numbers in the Number Line



Extend the range of normal floating point numbers.

Double Precision Numbers

Field	Size(bits)
S	1
E	11
M	52

- Approximate range of doubles
  - $\pm\ 2^{1023} = \pm\ 10^{308}$
  - This is a lot !!!

Floating Point Mathematics

```
A = 2^(50);
B = 2^(10);
C = (B+A)- A;
```

- C will be computed to be 0
  - There is no way of representing A+B in the IEEE 754 format
- A smart compiler can reorder the operations to increase precision
- Floating point math is approximate

Outline

- Boolean Algebra
- Positive Integers
- Negative Integers
- Floating Point Numbers
- Strings

ASCII Character Set

* ASCII – American Standard Code for Information Interchange
* It has 128 characters
* First 32 characters (control operations)
  * backspace (8)
  * line feed (10)
  * escape (27)
* Each character is encoded using 7 bits

ASCII Character Set

Character	Code	Character	Code	Character	Code
a	97	A	65	0	48
b	98	B	66	1	49
c	99	C	67	2	50
d	100	D	68	3	51
e	101	E	69	4	52
f	102	F	70	5	53
g	103	G	71	6	54
h	104	H	72	7	55
i	105	I	73	8	56
j	106	J	74	9	57
k	107	K	75	!	33
l	108	L	76	#	35
m	109	M	77	$	36
n	110	N	78	%	37
o	111	O	79	&	38
p	112	P	80	(	40
q	113	Q	81	)	41
r	114	R	82	*	42
s	115	S	83	+	43
t	116	T	84	,	44
u	117	U	85	.	46
v	118	V	86	;	59
w	119	W	87	=	61
x	120	X	88	?	63
y	121	Y	89	@	64
z	122	Z	90	^	94

Unicode Format

- UTF-8 (Universal character set Transformation Format)
  - UTF-8 encodes 1,112,064 characters defined in the Unicode character set. It uses 1-6 bytes for this purpose. E.g.अ आ क ख, ஞ□ㅂஜல
  - UTF-8 is compatible with ASCII. The first 128 characters in UTF-8 correspond to the ASCII characters. When using ASCII characters, UTF-8 requires just one byte. It has a leading 0.
  - Most of the languages that use variants of the Roman script such as French, German, and Spanish require 2 bytes in UTF-8. Greek, Russian (Cyrillic), Hebrew, and Arabic, also require 2 bytes.

UTF-16 and 32

- Unicode is a standard across all browsers and operating systems
- UTF-8 has been superseded by UTF-16, and UTF-32
- UTF-16 uses 2 byte or 4 byte encodings (Java and Windows)
- UTF-32 uses 4 bytes for every character (rarely used)

THE END