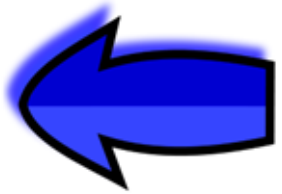
Chapter 2: The Language of Bits

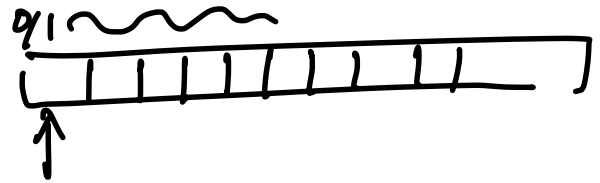
Basic Computer Architecture

Outline

* Boolean Algebra 
* Positive Integers
* Negative Integers
* Floating-Point Numbers
* Strings

What does a Computer Understand ?

* Computers do not understand natural human languages, nor programming languages
* They only understand the language of bits



Bit	0 or 1
Byte	8 bits
Word	4 bytes
kiloByte	1024 bytes
megaByte	10^6 bytes

1 KG = 10^3 gm

1 kB = 2^{10} bits = 1024

= 10^3 bit = 1000

1024 bytes => 2^{10} 2^{10}

10^6 bytes => $10^3 \times 10^3 = 10^6$

$2^{10} \times 2^{10}$

3

Review of Logical Operations

* A + B (A or B)

OR

A OR B

A || B

A | B

A	B	A + B
0	0	0
1	0	1
0	1	1
1	1	1

0/1

= Truth Table

* A.B (A and B)

AND

A B

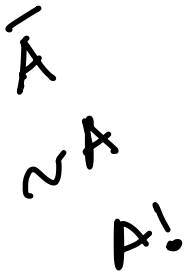
A	B	A.B
0	0	0
1	0	0
0	1	0
1	1	1

NOT

Review of Logical Operations - II

A	B	A NAND B
0	0	1
1	0	1
0	1	1
1	1	0

A	B	A NOR B
0	0	1
1	0	0
0	1	0
1	1	0



* NAND and NOR operations
* These are universal operations. They can be used to implement any Boolean function.

Review of Logical Operations

* XOR Operation : $(A \oplus B)$

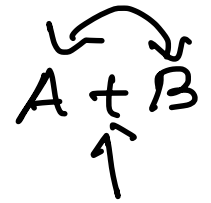
A	B	A XOR B
0	0	0
1	0	1
0	1	1
1	1	0

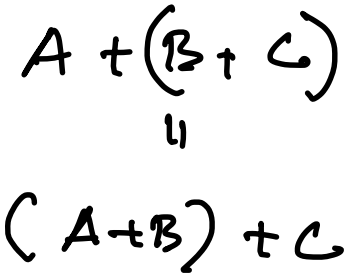
How many truth tables we can build?

Review of Logical Operations

* NOT operator
  * Definition: $\overline{0} = 1$, and $\overline{1} = 0$
  * Double negation: $\overline{\overline{A}} = A$, NOT of (NOT of A) is equal to A itself
* OR and AND operators
  * Identity: $A + 0 = A$, and $A.1 = A$
  * Annulment: $A + 1 = 1$, $A.0 = 0$

* Idempotence: A + A = A, A.A = A, The result of computing the OR and AND of A with itself is A.
* Complementarity: $A + \overline{A} = 1$, $A.\overline{A} = 0$
* Commutativity: A + B = B + A, A.B = B.A, the order of Boolean variables does not matter
* Associativity: A+(B+C) = (A+B)+C, A.(B.C) = (A.B).C, similar to addition and multiplication.
* Distributivity: A.(B + C) = A.B + A.C, A+ (B.C) = (A+B). (A+C) → Use this law to open up parantheses and simplify expressions





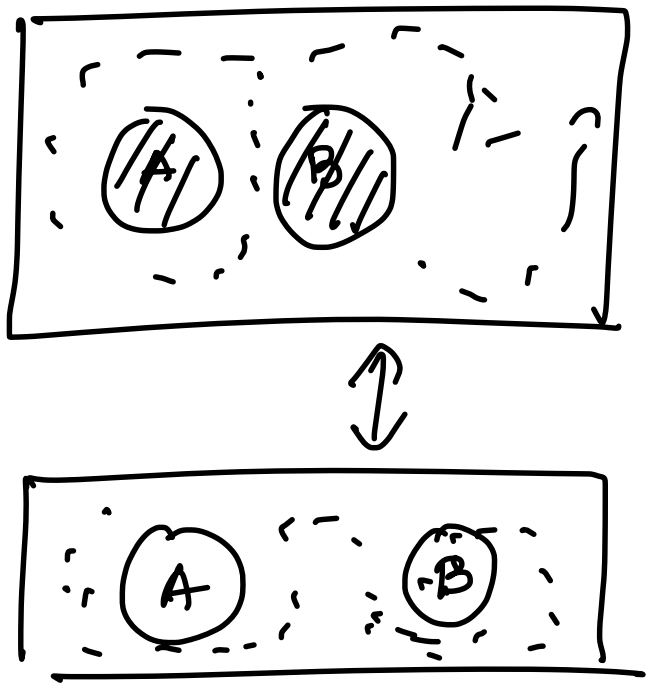
$x(y+z) = xy + xz$

De Morgan's Laws

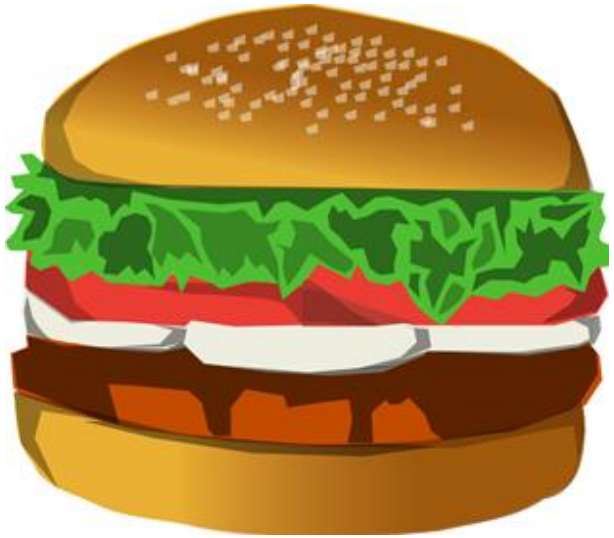
* Two very useful rules

$$\overline{A + B} = \overline{A}.\overline{B}$$

$$\overline{A.B} = \overline{A} + \overline{B}$$



Consensus Theorem



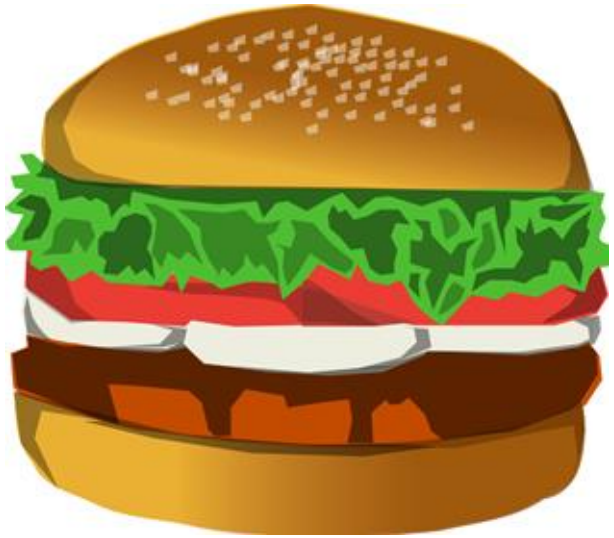
* Prove :
  * $X.Y + X.Z + Y.Z = X.Y + \bar{X}.Z$

$X.Y + \bar{X}Z + YZ(X + \bar{X})$

$XY + \bar{X}Z + XYZ + \bar{X}YZ$

$XY(1+Z) + \bar{X}Z(1+Y) = XY + \bar{X}Z$

Consensus Theorem



* Prove :
  * $X.Y + \bar{X}.Z + Y.Z = X.Y + \bar{X}.Z$

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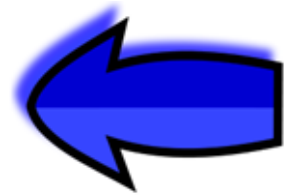
* Boolean Algebra
* Positive Integers
* Negative Integers
* Floating Point Numbers
* Strings



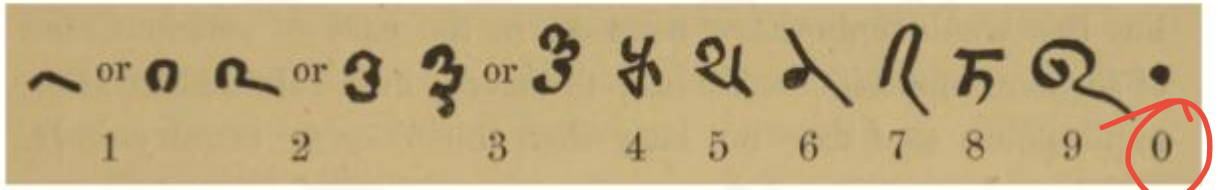
Representing Positive Integers

* Ancient Roman System

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

* Issues :
  * There was no notion of 0
  * Very difficult to represent large numbers
  * Addition, and subtraction (very difficult)

Indian System (place –value system)



Bakshali numerals, 7th century AD

* Uses the place value system

$5301 = 5 * 10^3 + 3 * 10^2 + 0 * 10^1 + 1*10^0$

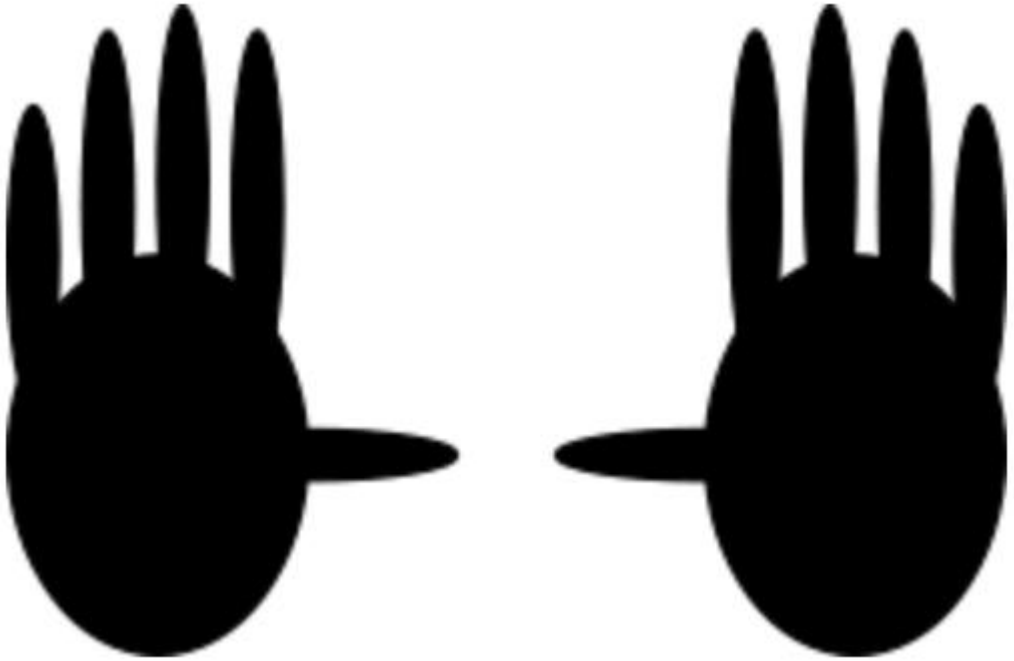
Example in base 10

5301 = place value.

base value

Number Systems in Other Bases

* Why do we use base 10 ?
  * because …



What if we had a world in which …

* People had only two fingers.

Binary Number System

* They would use a number system with base 2.

Number in decimal	Number in binary
5	101
100	1100100
500	111110100
1024	10000000000

19
= 10011
MSB
LSB
Least significant bit

2|5
2|2 — 1
1 — 0
5 = 101

$= 10^0 \times 5$

$2^2 \times 1 + 2^1 \times 0 + 2^0 \times 1$

base value = 2

place value starts from 0

MSB and LSB

* MSB (Most Significant Bit) → The leftmost bit of a binary number. E.g., MSB of 1110 is 1
* LSB (Least Significant Bit) → The rightmost bit of a binary number. E.g., LSB of 1110 is 0

Hexadecimal and Octal Numbers

* Hexadecimal numbers
  * Base 16 numbers – 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F
  * Start with 0x
* Octal Numbers = base = 8
  * Base 8 numbers – 0,1,2,3,4,5,6,7
  * Start with 0

Base = 16

0
1
2
3
4
5
6
7
8
9
– A — 10
– B — 11
– C — 12
– D — 13
– E — 14
– F — 15

1 Why Octal given that we have Hexa ?

Examples

Convert 110010111 to the octal format : $\underbrace{110}_{6}\ \underbrace{010}_{2}\ \underbrace{111}_{7} = 0627$

Convert 111000101111 to the hex format : $\underbrace{1110}_{E}\ \underbrace{0010}_{2}\ \underbrace{1111}_{F} = 0xE2F$

$a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7$

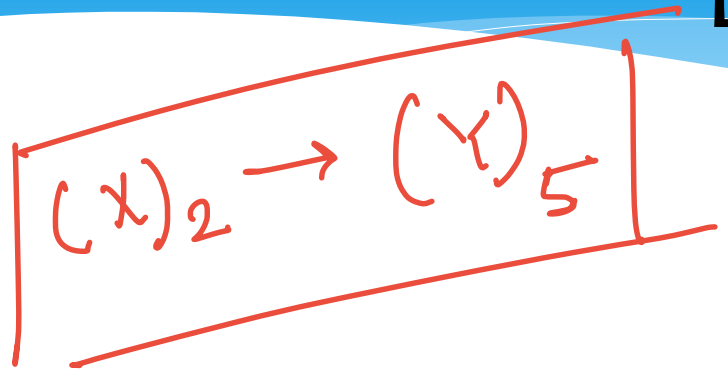
$N = (a_0 \times 2^0 + a_1 \times 2^1 + a_2 \times 2^2) + (a_3 2^3 + a_4 2^4 + a_5 2^5) + \cdots$

$= b_0 + 2^3 (a_3 \times 2^0 + a_4 \times 2^1 + a_5 2^2) + 2^6 (a_6 \times 2^0 + a_7 2^1 + a_8 2^2) + \cdots$

$= 2^0 (0-7) + 2^3 (0-7) + (2^3)^2 + (0-7) + \cdots$

$= 8^0 b_0 + 8^1 b_1 + 8^2 b_2 + \cdots$

Examples



Convert 110010111 to the octal format : $\underbrace{110}\ \underbrace{010}\ \underbrace{111}$ = 0627

Convert 111000101111 to the hex format : $\underbrace{1110}\ \underbrace{0010}\ \underbrace{1111}$ = 0xE2F

2. Can we convert a 2^n - base system to a base system which can't be represented by 2^m ; e.g.; binary $\rightarrow (Y)_5$

without converting to decimal

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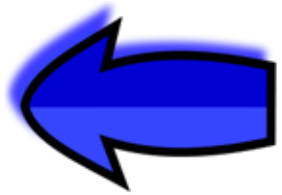
* Negative Integers 
* Floating Point Numbers
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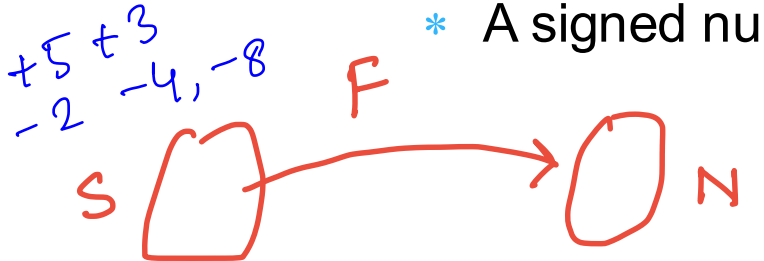
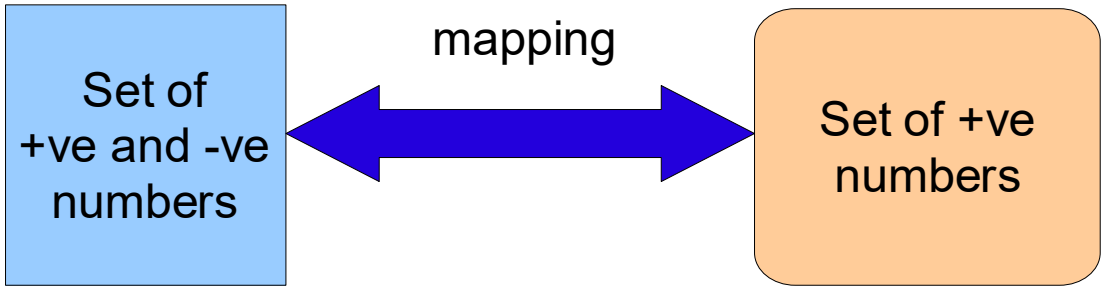
Representing Negative Integers

- Problem
  - Assign a binary representation to a negative integer
  - Consider a negative integer, S
  - Let its binary representation be : $x_n x_{n-1} \ldots x_2 x_1$ (x_i=0/1)
  - We can also expand it to represent an unsigned, +ve, number, N
  - If we interpret the binary sequence as :
    - An unsigned number, we get N
    - A signed number, we get S

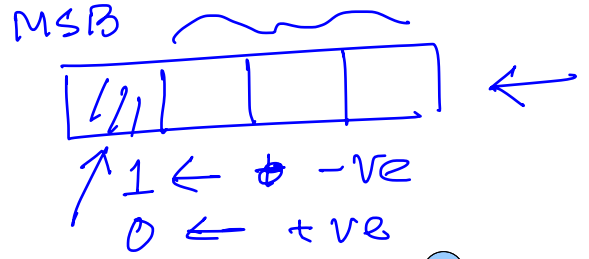


* We need a mapping :
  * F : S → N (mapping function)
  * S → set of numbers (both positive and negative – signed)
  * N → set of positive numbers (unsigned)



Properties of the Mapping Function

* Preferably, needs to be a one to one mapping
* All the entries in the set, S, need to be mapped
* It should be easy to perform addition and subtraction operations on the representation of signed numbers
* Assume an n bit number system



$$\text{SgnBit}(u) = \begin{cases} 1 , & u < 0 \\ 0 , & u >= 0 \end{cases}$$

Sign-Magnitude Base Representation

$$F(u) = SgnBit(u) * 2^{n-1} + |u|$$

0/1

sign bit	|u|

-5 n = 4 n-1 = 3

1 1 0 1

1 x 2^3 |5| 13

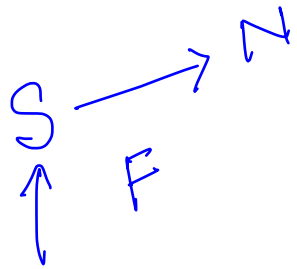
* Examples :
  * -5 in a 4 bit number system : 1101
  * 5 in a 4 bit number system : 0101
  * -3 in a 4 bit number system : 1011

+4 0 1 0 0 = 4

Problems

* There are two representations for 0
  * 000000
  * 100000
* Addition and subtraction are difficult
* The most important takeaway point :
  * Notion of the sign bit

1's Complement Representation



$$F(u) = \begin{cases} u, u \geq 0 \\ \sim(|u|) or (2^n - 1 - |u|), u < 0 \end{cases}$$

$n = 4$

$2^4 = 16$

$2^4 - 1 = 15$

- Examples in a 4 bit number system
  - 3 → 0011
  - -3 → 1100
  - 5 → 0101
  - -5 → 1010

$+3 = 0011$

$-3 = \sim(+3)$

$= 1100$

$= 12$

Notion of sign bit also exists

$F(-3) = 2^4 - 1 - |-3|$

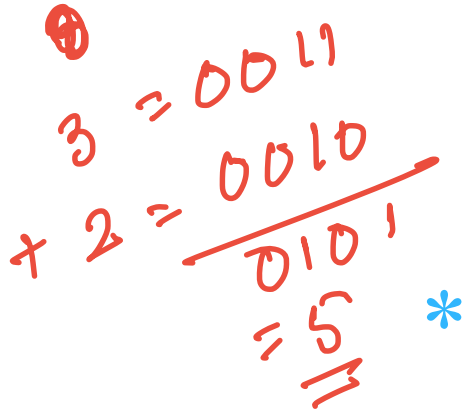
$= 15 - 3$

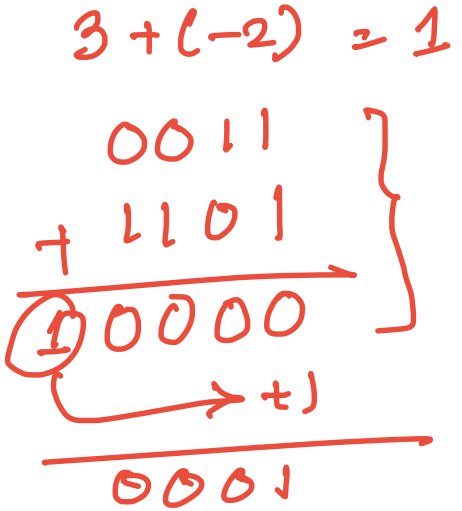
$= 12$

$= 1100$

Problems

* Two representations for 0
  * 0000000
  * 1111111
* Easy to add +ve numbers
* Hard to add -ve numbers
* Point to note :
  * The idea of a complement





Bias Based Approach

$$F(u) = u + bias$$

bias = 7

* Consider a 4 bit number system with bias equal to 7
  * -3 → 0100
  * 3 → 1010
* F(u+v) = F(u) + F(v) – bias
* Add and Sub are also easy
* Multiplication is difficult

$-3 = -3+7 = +4$

$+2 = +2+7 = +9$

$+3 = +3+7 = 10$ (1010)

$-3 = -3+7 = 4$ (0100)

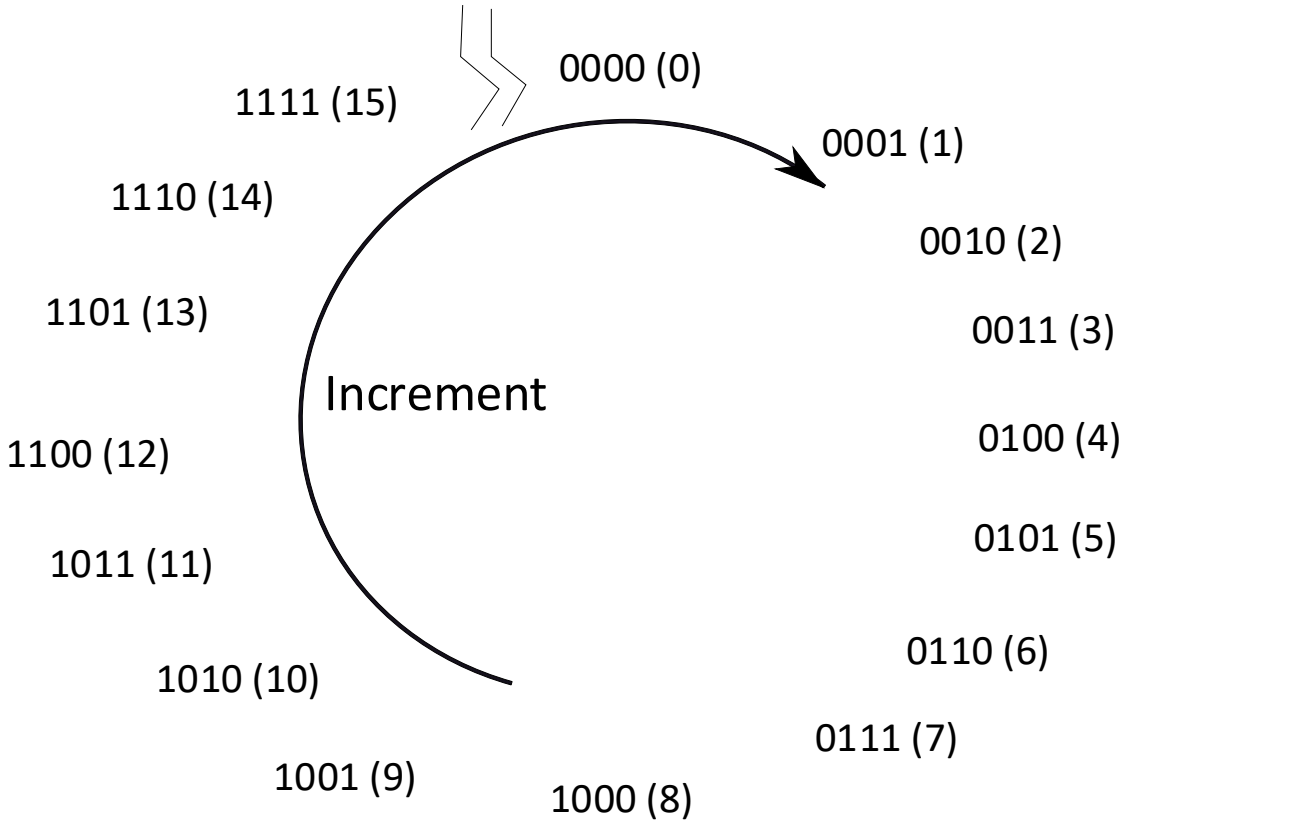
$1010 - 0100 = 1110$

$14 - 7 = 7$

$F(u-v) = F(u) - F(v) + bias.$

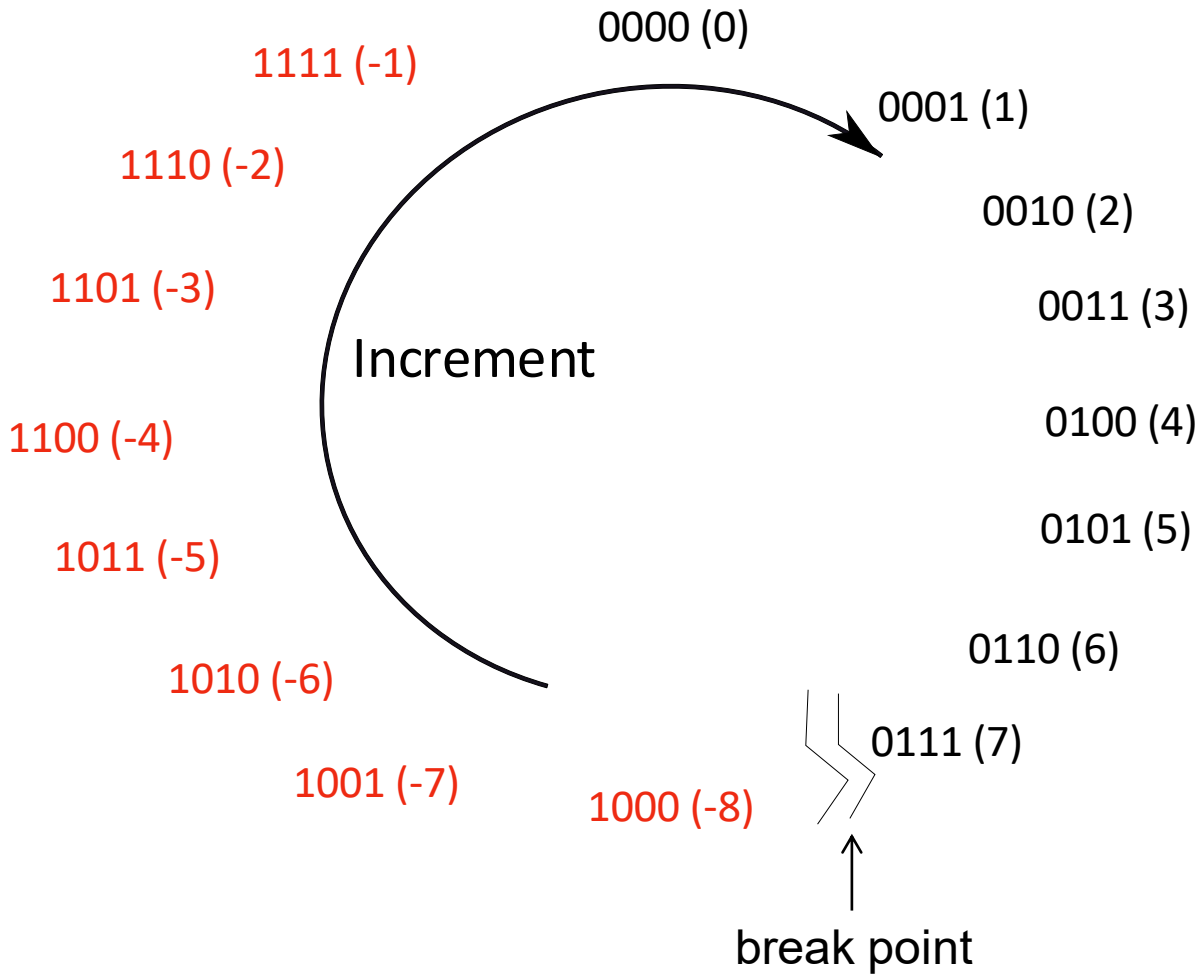
$F(3-3) = F(3) - F(3) + 7 = 7 = 0$

The Number Circle



Clockwise: increment
Anti-clockwise: decrement

Number Circle with Negative Numbers



Using the Number Circle

- To add M to a number, N
  - locate N on the number circle
  - If M is +ve
    - Move M steps clockwise
  - If M is -ve
    - Move M steps anti-clockwise, or 2^n – M steps clockwise
  - If we cross the break-point
    - We have an overflow
    - The number is too large/ too small to be represented

2's Complement Notation

$$F(u) = \begin{cases} u, 0 \le u \le 2^{n-1} - 1 \\ 2^n - |u|, -2^{n-1} \le u < 0 \end{cases}$$

* F(u) is the index of a point on the number circle. It varies from 0 to 2^n - 1
* Examples
  * 4 → 0100
  * -4 → 1100
  * 5 → 0101
  * -3 → 1101

Properties of the 2's Complement Notation

* Range of the number system :
  * $-2^{(n-1)}$ to $2^{n-1} - 1$
* There is a unique representation for 0
  → 000000
* msb of F(u) is equal to SgnBit(u)
  * Refer to the number circle
  * For a +ve number, $F(u) < 2^{(n-1)}$. MSB = 0
  * For a -ve number, $F(u) >= 2^{(n-1)}$. MSB = 1

Properties - II

* Every number in the range $[-2^{(n-1)}, 2^{(n-1)} - 1]$
  * Has a unique mapping
  * Unique point in the number circle
* $a \equiv b \rightarrow (a = b \bmod 2^n)$
* $\equiv$ means same point on the number circle
* $F(-u) \equiv 2^n - F(u)$
  * Moving F(u) steps counter clock wise is the same as moving $2^n - F(u)$ steps clockwise from 0

Prove : F(u+v) ≡ F(u) + F(v)

* Start at point u
  * Its index is F(u)
  * If v is +ve,
    * move v points clockwise. We arrive at F(u+v).
    * Its index is equal to (F(u) + v) mod 2^n.
    * Since v = F(v), we have F(u+v) = (F(u) + F(v)) mod 2^n

Prove : $F(u+v) \equiv F(u) + F(v)$

- If v is -ve,
  - move |v| points anti-clockwise.
  - Same as moving $2^n - |v|$ points clockwise.
  - We arrive at F(u+v).
  - $F(v) = 2^n - |v|$
  - The index – F(u+v) – is equal to:
    - $(F(u) + 2^n - |v|) \bmod 2^n = (F(u) + F(v)) \bmod 2^n$

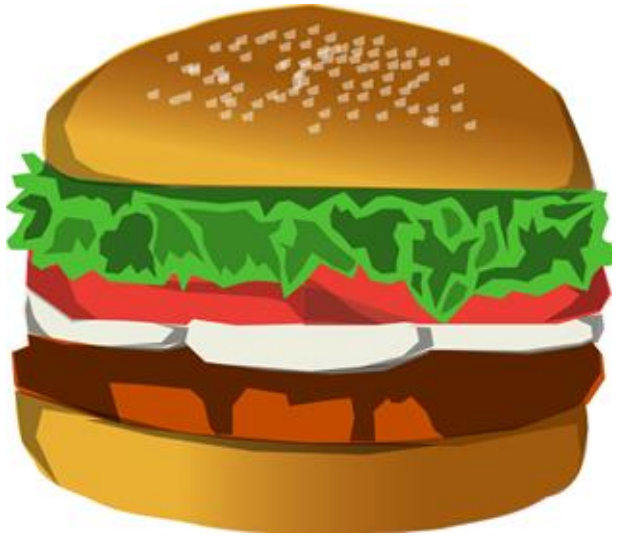
Subtraction

* $F(u-v) \equiv F(u) + F(-v)$

  $\equiv F(u) + 2^n - F(v)$

* Subtraction is the same as addition
* Compute the 2's complement of F(v)

Prove that :



* Prove that :

$$F(u*v) \equiv F(u) * F(v)$$

Computing the 2's Complement

* $2^n - u$

  $= 2^n - 1 - u + 1$

  $= \sim u + 1$

  * $\sim u$ (1's complement)

* 1's complement of 0100

$$\begin{array}{r} 1111 \\ -\ 0100 \\ \hline 1011 \end{array}$$

2's complement of 0100

$$\begin{array}{r} 1011 \\ +\ 0001 \\ \hline 1100 \end{array}$$

Sign Extension

* Convert a n bit number to a m bit 2's complement number (m > n)
* +ve
  * Add (m-n) 0s in the msb positions
  * Example, convert 0100 to 8 bits → 0000 0100
* -ve
  * F(u) = 2^n – |u| (n bit number) system
  * Need to calculate F'(u) = 2^m -|u|

Sign Extension - II

* $2^m - u - (2^n - u)$

  $= 2^m - 2^n$

  $= 2^n + 2^{(n+1)} + \ldots + 2^{(m-1)}$

  $= 11110000$

  m-n | n

$$F'(u) = F(u) + 2^m - 2^n$$

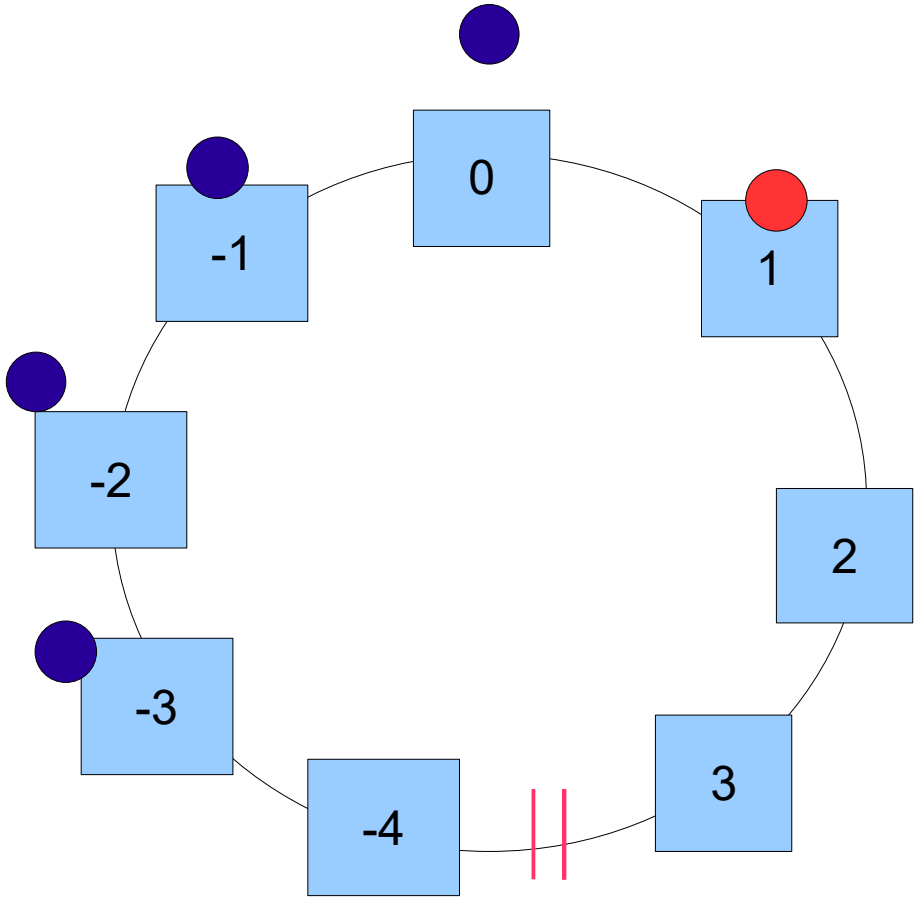
Sign Extension - III

* To convert a negative number :
  * Add (m-n) 1s in the msb positions
* In both cases, extend the sign bit by :
  * (m-n) positions

The Overflow Theorem

* Add : u + v
* If uv < 0, there will never be an overflow
* Let us go back to the number circle
  * There is an overflow only when we cross the break-point
* If uv = 0, one of the numbers is 0 (no overflow)
* If uv > 0, an overflow is possible

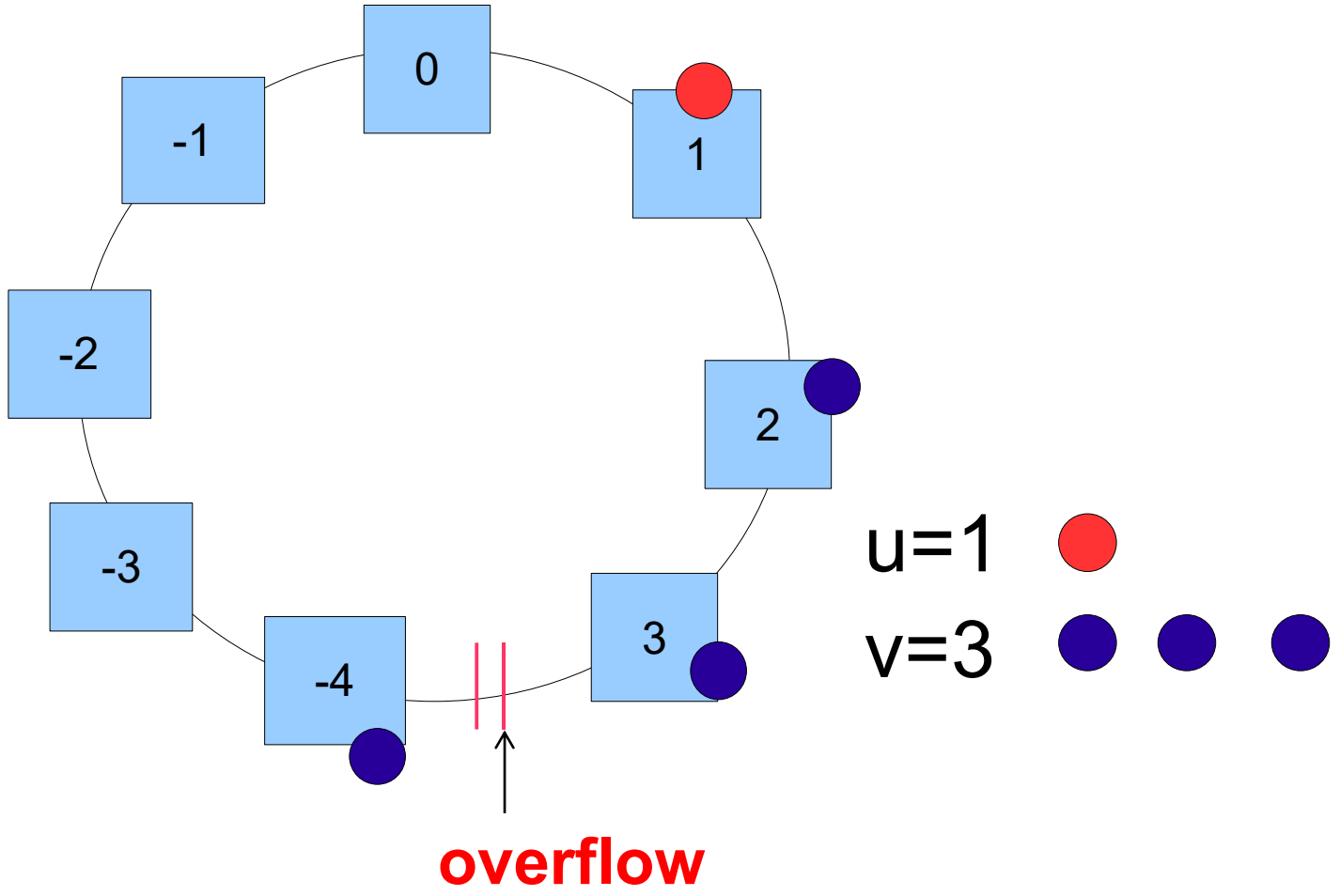
Number Circle: uv < 0



u=1

v=-4

Number Circle: uv > 0



Conditions for an Overflow

* uv <= 0
  * Never
* uv > 0 (u and v have the same sign)
  * The sign of the result is different from the sign of u

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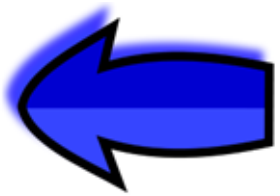
* Floating-Point Numbers 
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Floating-Point Numbers

* What is a floating-point number ?
  * 2.356
  * 1.3e-10
  * -2.3e+5
* What is a fixed-point number ?
  * Number of digits after the decimal point is fixed
  * 3.29, -1.83

Generic Form for Positive Numbers

* Generic form of a number in base 10

$$A = \sum_{i=-n}^{n} x_i 10^i$$

* Example :
  * $3.29 = 3 * 10^0 + 2*10^{-1} + 9*10^{-2}$

Generic Form in Base 2

* Generic form of a number in base 2

$$A = \sum_{i=-n}^{n} x_i 2^i$$

Number	Expansion
0.375	$2^{-2} + 2^{-3}$
1	2^{0}
1.5	$2^{0} + 2^{-1}$
2.75	$2^{1} + 2^{-1} + 2^{-2}$
17.625	$2^{4} + 2^{0} + 2^{-1} + 2^{-3}$

Binary Representation

* Take the base 2 representation of a floating-point (FP) number
* Each coefficient is a binary digit

Number	Expansion	BinaryRepresentation
0.375	$2^{-2} + 2^{-3}$	0.011
1	2^{0}	1.0
1.5	$2^{0} + 2^{-1}$	1.1
2.75	$2^{1} + 2^{-1} + 2^{-2}$	10.11
17.625	$2^{4} + 2^{0} + 2^{-1} + 2^{-3}$	10001.101

Normalized Form

* Let us create a standard form of all floating point numbers

$$A = (-1)^S * P * 2^X, (P = 1 + M, 0 \leq M < 1, X \square Z)$$

* S → sign bit, P → significand
* M → mantissa, X → exponent, **Z** → set of integers

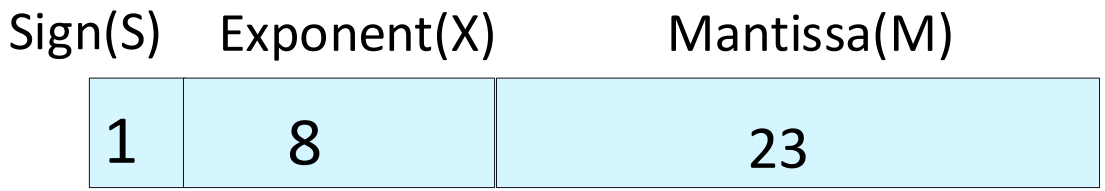
Examples (in decimal)

* 1.3827 * 1e-23
  * Significand (P) = 1.3827
  * Mantissa (M) = 0.3827
  * Exponent (X) = -23
  * Sign (S) = 0
* -1.2*1e+5
  * P = 1.2 , M = 0.2
  * S = 1, X = 5

IEEE 754 Format

* General Principles
  * The significand is of the form : 1.xxxxx
  * No need to waste 1 bit representing (1.) in the significand
  * We can just save the mantissa bits
  * Need to also store the sign bit (S), exponent (X)

IEEE 754 Format - II



* sign bit – 0 (+ve), 1 (-ve)
* exponent, 8 bits
* mantissa, 23 bits

Representation of the Exponent

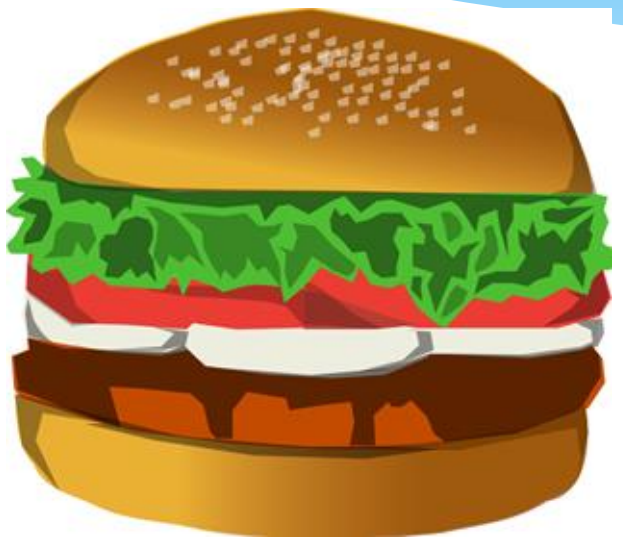
* Biased representation
  * bias = 127
  * E = X + bias
* Range of the exponent
  * 0 – 255 ⟷ -127 to +128
* Examples :
  * X = 0, E = 127
  * X = -23, E = 104
  * X = 30 , E = 157

Normal FP Numbers

* Have an exponent between -126 and +127
* Let us leave the exponents : -127, and +128 for special purposes.

$$A = (-1)^S * P * 2^{E-bias}$$

$$(P = 1 + M, 0 \leq M < 1, X \square Z, 1 \leq E \leq 254)$$



* What is the largest +ve normal FP number ?
* What is the smallest –ve normal FP number ?

Special Floating Point Numbers

E	M	Value
255	0	∞ if $S = 0$
255	0	$-\infty$ if $S = 1$
255	$\neq 0$	NAN(Not a number)
0	0	0
0	$\neq 0$	Denormal number

* NAN + x= NAN 1/0 = ∞
* 0/0 = NAN -1/0 = -∞
* $\sin^{-1}(5)$ = NAN

Denormal Numbers

```
f = 2^(-126);
g = f/2;
if (g == 0)
  print ("error");
```

* Should this code print "error" ?
* How to stop this behaviour ?

Denormal Numbers - II

$$A = (-1)^S * P * 2^{-126}$$

$$(P = 0 + M, 0 \leq M < 1)$$

* Significand is of the form : 0.xxxx
* E = 0, X = -126 (why not -127?)
* Smallest +ve normal number : 2^{-126}
* Largest denormal number :
  * $0.11...11 * 2^{-126} = (1 - 2^{-23}) * 2^{-126}$
    * $= 2^{-126} - 2^{-149}$

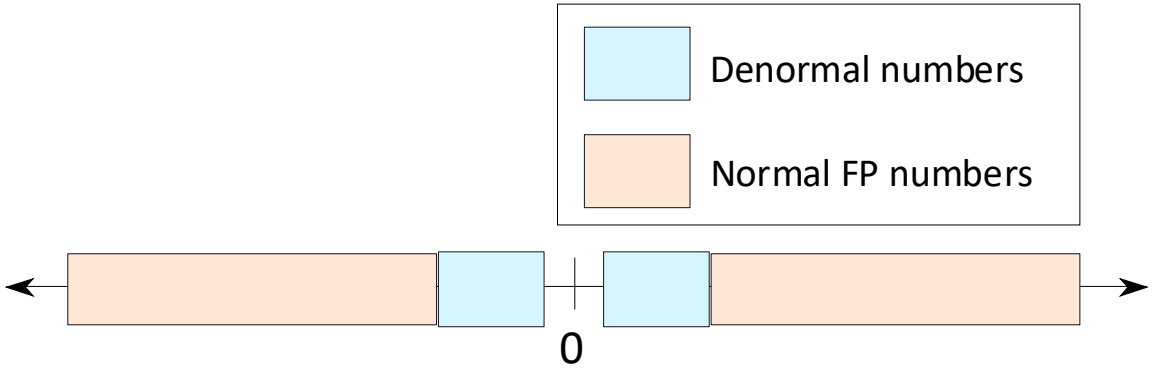
Example

Find the ranges of denormal numbers.

Answer

- For positive denormal numbers, the range is $[2^{-149}, 2^{-126} - 2^{-149}]$
- For negative denormal numbers, the range is $[-2^{-149}, -2^{-126} + 2^{-149}]$

Denormal Numbers in the Number Line



Extend the range of normal floating point numbers.

Double Precision Numbers

Field	Size(bits)
S	1
E	11
M	52

- Approximate range of doubles
  - $\pm 2^{1023} = \pm 10^{308}$
  - This is a lot !!!

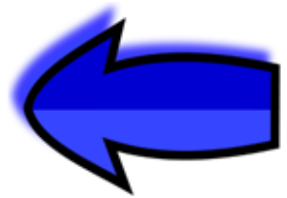
Floating Point Mathematics

```
A = 2^(50);
B = 2^(10);
C = (B+A)- A;
```

* C will be computed to be 0
  * There is no way of representing A+B in the IEEE 754 format
* A smart compiler can reorder the operations to increase precision
* Floating point math is approximate

Outline

* Boolean Algebra
* Positive Integers
* Negative Integers
* Floating Point Numbers
* Strings



ASCII Character Set

* ASCII – American Standard Code for Information Interchange
* It has 128 characters
* First 32 characters (control operations)
  * backspace (8)
  * line feed (10)
  * escape (27)
* Each character is encoded using 7 bits

ASCII Character Set

Character	Code	Character	Code	Character	Code
a	97	A	65	0	48
b	98	B	66	1	49
c	99	C	67	2	50
d	100	D	68	3	51
e	101	E	69	4	52
f	102	F	70	5	53
g	103	G	71	6	54
h	104	H	72	7	55
i	105	I	73	8	56
j	106	J	74	9	57
k	107	K	75	!	33
l	108	L	76	#	35
m	109	M	77	$	36
n	110	N	78	%	37
o	111	O	79	&	38
p	112	P	80	(	40
q	113	Q	81	)	41
r	114	R	82	*	42
s	115	S	83	+	43
t	116	T	84	,	44
u	117	U	85	.	46
v	118	V	86	;	59
w	119	W	87	=	61
x	120	X	88	?	63
y	121	Y	89	@	64
z	122	Z	90	^	94

Unicode Format

* UTF-8 (Universal character set Transformation Format)
  * UTF-8 encodes 1,112,064 characters defined in the Unicode character set. It uses 1-6 bytes for this purpose. E.g.अ आ क ख, தமஜஸ
  * UTF-8 is compatible with ASCII. The first 128 characters in UTF-8 correspond to the ASCII characters. When using ASCII characters, UTF-8 requires just one byte. It has a leading 0.
  * Most of the languages that use variants of the Roman script such as French, German, and Spanish require 2 bytes in UTF-8. Greek, Russian (Cyrillic), Hebrew, and Arabic, also require 2 bytes.

UTF-16 and 32

* Unicode is a standard across all browsers and operating systems
* UTF-8 has been superseded by UTF-16, and UTF-32
* UTF-16 uses 2 byte or 4 byte encodings (Java and Windows)
* UTF-32 uses 4 bytes for every character (rarely used)

THE END